Around the Van Daele–Schmüdgen Theorem

Yury Arlinskiĭ, Valentin A. Zagrebnov

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YURY ARLINSKIĬ AND VALENTIN A. ZAGREBNOV

Abstract. For a *bounded* non-negative self-adjoint operator acting in a complex, infinite-dimensional, separable Hilbert space $\mathcal{H}$ and possessing a dense range $\mathcal{R}$ we propose a new approach to characterisation of phenomenon concerning the existence of subspaces $\mathfrak{M}\subset\mathcal{H}$ such that $\mathfrak{M}\cap\mathcal{R}=\mathfrak{M}^\perp\cap\mathcal{R}=\{0\}$. We show how the existence of such subspaces leads to various *pathological* properties of *unbounded* self-adjoint operators related to von Neumann theorems [31]–[33]. We revise the von Neumann-Van Daele-Schmüdgen assertions [31], [39], [36] to refine them. We also develop a new systematic approach, which allows to construct for any *unbounded* densely defined symmetric/self-adjoint operator T *infinitely* many pairs $\langle T_1,T_2\rangle$ of its closed densely defined restrictions $T_k\subset T$ such that $\operatorname{dom}(T^*T_k)=\{0\}$ ($\Rightarrow \operatorname{dom}T_k^2=\{0\}$) $k=1,2$ and $\operatorname{dom}T_1\cap\operatorname{dom}T_2=\{0\}$, $\operatorname{dom}T_1\dotplus\operatorname{dom}T_2=\operatorname{dom}T$.

Contents

1. Introduction

Throughout this paper we consider infinite-dimensional and separable Hilbert spaces over the field $\mathbb{C}$ of complex numbers. If $\mathcal{H}$ is a Hilbert space, then its (proper) linear subset $\mathfrak{M}\subset\mathcal{H}$ is called a *linear manifold*. The closure $\overline{\mathfrak{M}}$ in topology of $\mathcal{H}$ is itself a Hilbert space. We call this *closed* linear manifold a *subspace* of the space $\mathcal{H}$. Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be

2010 *Mathematics Subject Classification.* 47A05, 47A07, 47A20, 47A64, 47B25.
Key words and phrases. Operator range, von Neumann theorem, Van Daele–Schmüdgen theorem, parallel addition, shorted operator, lifting, Friedrichs extension, Kreĭn extension.

linear manifolds of $\mathcal{H}$. Then $\mathfrak{M}_1 + \mathfrak{M}_2$ denotes the *sum* of manifolds, which is the smallest linear manifold that contains $\mathfrak{M}_1$ and $\mathfrak{M}_2$. If intersection of subsets $\mathfrak{M}_1$ and $\mathfrak{M}_2$ has only *zero vector* in common, we denote the sum by $\mathfrak{M}_1 \dot{+} \mathfrak{M}_2$ and call it the *direct sum* of linear manifolds. If in addition these two linear manifolds are mutually orthogonal, then we denote their sum as $\mathfrak{M}_1 \oplus \mathfrak{M}_2$ and we call it the *orthogonal sum*. All these linear operations can be obviously extended to *subspaces* of $\mathcal{H}$. Note that the sum $\overline{\mathfrak{M}}_1 + \overline{\mathfrak{M}}_2$, or the direct sum $\overline{\mathfrak{M}}_1 \dot{+} \overline{\mathfrak{M}}_2$ of subspaces is not obligatory a subspace, but it is true for the orthogonal sum $\overline{\mathfrak{M}}_1 \oplus \overline{\mathfrak{M}}_2$.

We use the symbols $\operatorname{dom} T$, $\operatorname{ran} T$, $\ker T$ for manifolds which are respectively domain, range, and null-subspace of a linear operator T. The closures of two first manifolds are denoted by $\overline{\operatorname{dom}}\, T$, $\overline{\operatorname{ran}}\, T$. The identity operator in a Hilbert space $\mathcal{H}$ is denoted by $I := I_{\mathcal{H}}$. If $\mathfrak{L}$ is a subspace of $\mathcal{H}$, the orthogonal projection in $\mathcal{H}$ onto $\mathfrak{L}$ is denoted by $P_{\mathfrak{L}}$. By $\mathfrak{L}^{\perp}$ we denote the subspace which is the orthogonal complement of $\mathfrak{L}$, which is $\mathfrak{L}^{\perp} = \mathcal{H} \ominus \mathfrak{L}$. We use notation $T{\upharpoonright}\mathcal{N}$ for restriction of a linear operator T on the set $\mathcal{N} \subset \operatorname{dom} T$.

A linear operator $\mathcal{A}$ in a Hilbert space is called *non-negative* (or *positive*) if $(\mathcal{A}f, f) \geq 0$ for all $f \in \operatorname{dom}\mathcal{A}$ and it is called *positive definite* if $(\mathcal{A}f, f) \geq c\|f\|^2$ for some $c > 0$. We write $\mathcal{A} \geq 0$ if $\mathcal{A}$ is a non-negative operator. Then the natural order $\mathcal{A} \leq \mathcal{C}$ of two positive (bounded) self-adjoint operators is implied by $\mathcal{C} - \mathcal{A} \geq 0$.

The linear space of bounded operators from the Hilbert space $\mathcal{H}$ to the Hilbert space $\mathfrak{H}$ is denoted by $\mathbf{B}(\mathcal{H}, \mathfrak{H})$ and the Banach algebra $\mathbf{B}(\mathcal{H}, \mathcal{H})$ by $\mathbf{B}(\mathcal{H})$. The set of all bounded self-adjoint non-negative operators in $\mathcal{H}$ we denote by $\mathbf{B}^+(\mathcal{H})$. Then *non-singular* operators $\mathbf{B}^+_0(\mathcal{H}) \subset \mathbf{B}^+(\mathcal{H})$ is the subset of $\mathbf{B}^+(\mathcal{H})$ with $\ker B = \{0\}$. If $T : \mathcal{H} \to \mathfrak{H}$ is a closed linear operator in a Hilbert space $\mathcal{H}$, then we used to consider the linear manifold $\operatorname{dom} T$ as a Hilbert space with respect to the *graph inner product*:

$$(u, v)_T := (u, v)_{\mathcal{H}} + (Tu, Tv)_{\mathfrak{H}} \ .$$

Now we recall two results, which are established by A.Van Daele. The first result demonstrates some *pathological* properties of *unbounded* operators. It was inspired by the well-known (and somewhat surprising) J. von Neumann theorem [31], which states that for any unbounded self-adjoint operator $\mathcal{A}$ there is a unitary operator U such that $\operatorname{dom}\mathcal{A}$ and $\operatorname{dom} U^*\mathcal{A}U$ have only the *zero* vector in common.

Theorem 1.1. [39, Theorem 2.2]. *Let T be a positive self-adjoint operator in the Hilbert space $\mathcal{H}$. If $\ker T = \{0\}$ (non-singular operator), then there exists two densely defined closed symmetric restrictions S_1 and S_2 of T such that $\operatorname{dom} S_1 \cap \operatorname{dom} S_2 = \{0\}$.*

In fact one can see from the proof of this theorem that moreover: it is possible to choose the symmetric densely defined operators S_1 and S_2 in such a way that their ranges $\operatorname{ran} S_1$ and $\operatorname{ran} S_2$ are *orthogonal*. This second result was formulated by Van Daele in [40] as a corollary the following general assertion:

Theorem 1.2. [40, Proposition 3]. *Let B be a positive self-adjoint operator in the Hilbert space $\mathcal{H}$. Suppose that $\ker B = \{0\}$ and $\operatorname{ran} B \neq \mathcal{H}$, i.e. the inverse operator B^{-1} is unbounded. Then there exist two linear manifolds $\mathfrak{M}_1$ and $\mathfrak{M}_2$ of $\operatorname{dom} B$ such that:*
(i) $\mathfrak{M}_1 \perp \mathfrak{M}_2$,
(ii) *the direct sum $\mathfrak{M}_1 \dot{+} \mathfrak{M}_2$ is dense in $\mathcal{H}$* ,
(iii) *the linear manifolds $B\,\mathfrak{M}_1$ and $B\,\mathfrak{M}_2$ are also dense in $\mathcal{H}$.*

Remark 1.3. *Note that if unbounded operator T in Theorem 1.1 is boundedly invertible, then one can put $B := T^{-1}$ and apply Theorem 1.2 for $B \in \mathbf{B}_0^+(\mathcal{H})$. This means that one can find two orthogonal linear manifolds $\mathfrak{M}_1$, $\mathfrak{M}_2 \subset \mathcal{H}$ and define two symmetric operators S_1, S_2 with dense domains* $\operatorname{dom} S_1 := B\,\mathfrak{M}_1$, $\operatorname{dom} S_2 := H\,\mathfrak{M}_2$ *by restrictions*

$$S_1 := T\upharpoonright \operatorname{dom} S_1 \quad , \quad S_2 := T\upharpoonright \operatorname{dom} S_2 \ .$$

Then by construction of operators S_1 and S_2 the ranges $\operatorname{ran} S_1 = \mathfrak{M}_1$ *and* $\operatorname{ran} S_2 = \mathfrak{M}_2$ *are orthogonal.*

The next result is due to K.Schmüdgen. It was apparently motivated by [31] and by the arguments in [39] and [19]. In paper [36] Schmüdgen proved the following assertion.

Theorem 1.4. [36, Theorem 5.1] *Let H be a closed unbounded densely defined linear operator in the Hilbert space $\mathcal{H}$. Then there exists an orthogonal projection P such that*

$$P\mathcal{H} \cap \operatorname{dom} H = (I-P)\mathcal{H} \cap \operatorname{dom} H = \{0\} \ . \tag{1.1}$$

Remark 1.5. *In fact the statements formulated in Remark 1.3 and in Theorem 1.4 are equivalent in the case when $\mathfrak{M}_1 = \mathfrak{M}$ is a subspace, i.e. $\mathfrak{M}_2 = \mathfrak{M}^\perp = \mathcal{H} \ominus \mathfrak{M}$, and if $H \geq 0$ is a non-singular, unbounded, self-adjoint operator.*
Indeed, let $B \in \mathbf{B}_0^+(\mathcal{H})$ with $\operatorname{ran} B \neq \mathcal{H}$. *Then B is invertible and $T := B^{-1}$ is unbounded self-adjoint operator $T \geq 0$ with* $\operatorname{dom} T = \operatorname{ran} B$. *By Theorem 1.2(i) and by our assumption that $\mathfrak{M}_1 = \mathfrak{M}$, $\mathfrak{M}_2 = \mathcal{H} \ominus \mathfrak{M}$, Theorem 1.2(iii) yields: $(\overline{B\mathfrak{M}^\perp} = \mathcal{H}) \Leftrightarrow (\forall u \in \mathfrak{M}^\perp \wedge \phi \in \mathcal{H}, (Bu, \phi) = 0 \Leftrightarrow \phi = 0)$. Since B is self-adjoint, we have $(\forall u \in \mathfrak{M}^\perp \wedge \phi \in \mathcal{H}, (u, B\phi) = 0 \Leftrightarrow B\phi \in \mathfrak{M} \cap \operatorname{ran} B)$, and therefore $(\overline{B\mathfrak{M}^\perp} = \mathcal{H} \Leftrightarrow \mathfrak{M} \cap \operatorname{ran} B = \{0\})$. The same arguments yield $(\overline{B\mathfrak{M}} = \mathcal{H} \Leftrightarrow \mathfrak{M}^\perp \cap \operatorname{ran} B = \{0\})$. Hence, one gets*

$$\begin{aligned}\overline{B\mathfrak{M}^\perp} = \overline{B\mathfrak{M}} = \mathcal{H} \iff \mathfrak{M} \cap \operatorname{ran} B = \mathfrak{M}^\perp \cap \operatorname{ran} B = \{0\} \\ \iff \mathfrak{M} \cap \operatorname{dom} T = \mathfrak{M}^\perp \cap \operatorname{dom} T = \{0\} \ ,\end{aligned}$$

that gives (1.1) for $\mathfrak{M} = P\mathcal{H}$ and $T = H$.
On the other hand, let $H \geq 0$ be unbounded, self-adjoint operator, which is boundedly invertible: $H^{-1} = B$. Then by Theorem 1.4 there exists an orthogonal projector P such that

$$\begin{aligned}P\mathcal{H} \cap \operatorname{dom} H = (I-P)\mathcal{H} \cap \operatorname{dom} H = \{0\} \iff \mathfrak{M} \cap \operatorname{ran} B = \mathfrak{M}^\perp \cap \operatorname{ran} B = \{0\} \\ \iff \overline{B\mathfrak{M}} = \overline{B\mathfrak{M}^\perp} = \mathcal{H} \ ,\end{aligned}$$

where $\mathfrak{M} := P\mathcal{H}$ and $\mathfrak{M}^\perp = (I-P)\mathcal{H}$. This coincides with Remark 1.3 for $\mathfrak{M}_1 = \mathfrak{M}$ and $\mathfrak{M}_2 = \mathfrak{M}^\perp$.

Remark 1.6. *In the present paper we call the statements of Theorems 1.1-1.4 and of Remark 1.3 as the* Van Daele–Schmüdgen Theorem. *Our aim is to develop a new systematic approach to treat the pathologies of unbounded operators, which is motivated by this Theorem.*

Note that using Theorem 1.4 and the Cayley transformation Schmüdgen also proved in [36] an extended version of the Van Daele Theorem 1.1. It is related to the *domain triviality* problem of the square of symmetric operator. This problem was formulated and studied for the first time in [27], [28], [16], [13].

Theorem 1.7. [36, Theorem 5.2]. *For each unbounded self-adjoint operator H in $\mathcal{H}$ there exists closed densely defined restrictions of H to symmetric operators H_1 and H_2 such that*

$$\operatorname{dom} H_1 \cap \operatorname{dom} H_2 = \{0\} \quad \textit{and} \quad \operatorname{dom} H_1^2 = \operatorname{dom} H_2^2 = \{0\} \ . \tag{1.2}$$

Later, J.R.Brasche and H.Neidhardt [11] showed that this result remains true if the condition of self-adjoint operator H is replaced in Theorem 1.7 by a closed symmetric, but non-self-adjoint operator.

Note also that the first assertion in (1.2) was proved by Van Daele [39] under additional assumptions: $H \geq 0$ and $\ker H = \{0\}$, see Theorem 1.1.

Remark that original proofs of Theorems 1.1, 1.2, and 1.4 are essentially based on the *spectral decompositions* of self-adjoint operators and the theory of functions (Fourier series, analytic functions etc). In the present paper we first elucidate and then we give a new proof of the Van Daele–Schmüdgen theorem. The proof includes also a generalisation of this theorem. To this aim we use only operator methods. Our approach uses two key ingredients:

(1) The classical von Neumann theorem [31] (see also [32], [33]), which in particular states that for any unbounded self-adjoint operator A with a dense domain in $\mathcal{H}$ there exists a densely defined self-adjoint operator B such that intersection of their domains is trivial: $\operatorname{dom} A \cap \operatorname{dom} B = \{0\}$.
(2) The notion and properties of a *parallel addition* operation for two bounded non-negative self-adjoint operators [2],[3].

As we mentioned above the von Neumann theorem states in particular that for any *unbounded* self-adjoint operator H there exists a unitary U such that $\operatorname{dom} H \cap \operatorname{dom}(U^*HU) = \{0\}$. Then setting $J := 2P - I$ for projection P satisfying (1.1), we obtain as a corollary a refined version of this theorem: there exists a *unitary* and *self-adjoint* operator J such that $\operatorname{dom} H \cap \operatorname{dom}(JHJ) = \{0\}$, see Sections 3.1 and 3.3.

Our arguments allow to obtain more details about properties of restrictions of self-adjoint operators treated in Theorems 1.1-1.7 and to revise the Van Daele–Schmüdgen and the Brasche–Neidhardt theorems, see Section 3.4. We note also that the von Neumann theorem [31]-[33] and Schmüdgen's result [37] are related to results in [29], [30] about another kind of pathological properties of operators unbounded from above *and* from below. These papers solved the problem of existence of densely defined symmetric semi-bounded restrictions to *stability domains* of initially unbounded from *below* symmetric operators. The same theorems together with the operator *parallel addition* play essential role in [6] in order to construct counterexamples to some statements in [25] related to the Q-functions of Hermitian contractions.

Here is a brief review of contents of the paper. In Section 2 we recall some basic facts of the operator theory indispensable for formulations and proofs of our main results. They are: the operator ranges, the concept of parallel addition, the Kreĭn shorted operators, the self-adjoint extensions of non-negative operators, and few relevant fundamental statements like von Neumann's and Douglas' theorems.

Section 3 collects our main results. We start by Section 3.1, where different characterisations for trivial intersections of operator ranges with subspaces are presented. This preparation is aimed to describe then essential steps of our approach.

Our key statement (Theorem 3.7) is that for a given $A \in \mathbf{B}_0^+(\mathcal{H})$ with $\operatorname{ran} A \neq \mathcal{H}$, we can find a *continuum* set of different subspaces $\mathfrak{M} \subset \mathcal{H}$ satisfying

$$\mathfrak{M} \cap \operatorname{ran} A^{1/2} = \mathfrak{M}^\perp \cap \operatorname{ran} A^{1/2} = \{0\} \ . \tag{1.3}$$

In Theorem 3.9 we show the existence of *increasing* (*decreasing*) chains of subspaces possessing the trivial intersection property (1.3). To this aim we use the *lifting* of operator A. It is defined as a representation of A generated by orthogonal projection $P_{\mathfrak{M}} : \mathcal{H} \to \mathfrak{M}$, which has the form

$$A = T^{1/2} P_{\mathfrak{M}} T^{1/2} \ . \tag{1.4}$$

Here $T \in \mathbf{B}_0^+(\mathcal{H})$ is the sum $T = A + B$, where $B \in \mathbf{B}_0^+(\mathcal{H})$ with $\operatorname{ran} B^{1/2} \cap \operatorname{ran} A^{1/2} = \{0\}$. Note that by virtue of trivial intersection of $\operatorname{ran} B^{1/2}$ and $\operatorname{ran} A^{1/2}$, the subspaces $\mathfrak{M}$ and $\mathfrak{M}^\perp$ have trivial intersections with $\operatorname{ran} T^{1/2}$:

$$\mathfrak{M} \cap \operatorname{ran} T^{1/2} = \mathfrak{M}^\perp \cap \operatorname{ran} T^{1/2} = \{0\}. \tag{1.5}$$

In Section 3.2 we study the existence of the lifting in the form (1.4) with (1.5), when the subset $\mathfrak{M}$ possessing (1.3) is given. Then conditions on the entries of $A \in \mathbf{B}_0^+(\mathcal{H})$ in its block-operator matrix representation with respect to decomposition $\mathcal{H} = \mathfrak{M} \oplus \mathfrak{M}^\perp$ are found. We give examples that *not all* subspaces $\mathfrak{M}$ possessing (1.3) can be constructed applying a *general* form of the operator lifting (1.4). This indicates that our method is not exhaustive. It also means that the problem of construction of *all* subspaces $\mathfrak{M}$ verifying (1.3) for a given operator A is *open*.

Nevertheless, our method of the operator *lifting* allows to obtain more detailed information about the *hierarchy* of possible subspaces $\mathfrak{M}$ and to establish a number of new results about it. In particular, we prove that for a given $A \in \mathbf{B}_0^+(\mathcal{H})$ with $\operatorname{ran} A \neq \mathcal{H}$ there exists a one-parameter family of these subspaces, see Theorem 3.7 and Proposition 3.8, as well as some increasing (decreasing) infinite chains of subspaces $\mathfrak{M}$ with the property (1.3), see Theorem 3.9 and Corollary 3.12.

In Theorems 3.17, 3.20 of the next Section 3.3 we revise the Schmügen result (Theorem 1.4). Moreover, in Theorem 3.26 we construct decreasing/increasing families (in the sense of associated closed quadratic forms) of *pairs* of non-negative self-adjoint operators with trivial interactions of their form-domains with domain of a given unbounded non-negative self-adjoint operator. Then we investigate the limiting behaviour of their resolvents and of the corresponding one-parameter semigroups.

These results allow to scrutinise in Section 3.4 the *triviality domain problem* for products/powers of *unbounded* operators, cf Theorem 1.7. We propose a systematic method for construction of examples of pairs operators $\langle B, \widetilde{B} \rangle$ consisting of closed densely defined symmetric operator B and its symmetric/self-adjoint extension $\widetilde{B}$, such that $\operatorname{dom}(\widetilde{B}^* B) = \{0\}$. This gives abstract examples of symmetric operators B with trivial squares and allows us to refine the Van Daele–Schmüdgen and the Brasche–Neidhardt theorems. Under certain additional conditions we show in Theorems 3.31 and 3.32 that the products in different order, i.e., operators $B\widetilde{B}$ and $\widetilde{B}B$ are densely defined and we describe their Friedrichs and Kreĭn self-adjoint extensions.

2. Preliminaries

2.1. **Operator ranges.** Following [19] we call linear manifold $\mathcal{R}$ in a Hilbert space $\mathcal{H}$ an *operator range*, if it is the range of *some* bounded linear operator on $\mathcal{H}$. Note that even for bounded operators the operator ranges possess certain special features that distinguish them from *arbitrary* linear manifolds since their properties may be more pathological.

Clearly, if an operator range $\mathcal{R}$ is unclosed and dense in $\mathcal{H}$, then it is a domain of a non-negative self-adjoint unbounded operator in $\mathcal{H}$. Indeed, if $\mathcal{R} = \operatorname{ran}\mathcal{A}$, $\mathcal{A} \in \mathbf{B}(\mathcal{H})$, then $\mathcal{R} = \operatorname{ran}|\mathcal{A}^*|$, where $|\mathcal{A}^*| := (\mathcal{A}\mathcal{A}^*)^{1/2}$ is non-negative self-adjoint bounded operator. Since $\mathcal{R}$ is dense in $\mathcal{H}$ we get $\ker|\mathcal{A}^*| = \{0\}$. The inequality $\mathcal{R} \neq \mathcal{H}$ yields that the operator $T = |\mathcal{A}^*|^{-1}$ is unbounded non-negative self-adjoint operator and $\operatorname{dom} T = \mathcal{R}$. Conversely, if T is a non-negative unbounded closed and densely defined linear operator, then $\operatorname{dom} T = \operatorname{dom}|T|$, for $|T| = (T^*T)^{1/2}$. Consequently, one obtains

$$\operatorname{dom} T = \operatorname{ran}\,(|T| + I)^{-1}\ .$$

This means that $\operatorname{dom} T$ is an operator range. Various characterizations of operator ranges can be found in [19].

2.2. **The Douglas theorem.**

Theorem 2.1. [17] *For every $A, B \in \mathbf{B}(\mathcal{H})$ the following statements are equivalent:*

(i) $\operatorname{ran} A \subset \operatorname{ran} B$*;*
(ii) $A = BC$ *for some* $C \in \mathbf{B}(\mathcal{H})$*;*
(iii) $AA^* \le \lambda BB^*$ *for some* $\lambda \ge 0$.

Moreover, there is a unique operator C satisfying $\operatorname{ran} C \subset \overline{\operatorname{ran}}\, B^*$*, in which case* $\ker C = \ker A$.

The next relations follow from Theorem 2.1 (see [19], Sect.4):

$$\left(\sum_{j=1}^{n} F_j\right)^{1/2} = \operatorname{ran} F_1^{1/2} + \ldots + \operatorname{ran} F_n^{1/2},\ \{F_j\}_{j=1}^n \subset \mathbf{B}^+(\mathcal{H}), \tag{2.1}$$

$$\operatorname{ran}\,\left(F^{1/2} M F^{1/2}\right)^{1/2} = F^{1/2}\operatorname{ran} M^{1/2},\ F, M \in \mathbf{B}^+(\mathcal{H}). \tag{2.2}$$

This yields, in particular, that if $T_1, \ldots, T_j$ are closed and densely defined linear operators in $\mathcal{H}$, then the linear manifold

$$\operatorname{dom} T_1 + \ldots + \operatorname{dom} T_n$$

is domain of a closed linear operator.

2.3. **The von Neumann theorem.** In paper [31] (see also [32], [33]) John von Neumann established the following fundamental result:

Theorem 2.2. *For any unbounded self-adjoint operator H in a Hilbert space there exists a unitary operator U with the property*

$$\operatorname{dom} H \cap \operatorname{dom}(U^*HU) = \{0\}.$$

Special examples of self-adjoint operators A and B with $\operatorname{dom} A \cap \operatorname{dom} B = \{0\}$ one can find in [12], [39], [23].

In terms of operator ranges the statement of Theorem 2.2 takes the following form:

Theorem 2.3. *If $\mathcal{R}$ is a nonclosed and dense operator range in a Hilbert space $\mathcal{H}$, then there is a unitary operator U on $\mathcal{H}$ such that*

$$\mathcal{R} \cap U\mathcal{R} = \{0\}. \tag{2.3}$$

Corollary 2.4. *If $\mathcal{R}$ is a nonclosed operator range in a Hilbert space, then there exists a continuous one-parameter unitary group $\{U_t\}_{t\in\mathbb{R}}$ such that $U_s\mathcal{R} \cap U_t\mathcal{R} = \{0\}$ for $s \neq t$.*

The proof of Theorem 2.3 and Corollary 2.4 can be found in e.g. [15] and [19].

2.4. **The parallel sum of operators.** Let F and G be two bounded non-negative operators on $\mathcal{H}$. The *parallel sum* $F : G$ of F and G is defined by the quadratic form:

$$((F : G)h, h) := \inf_{f,g\in\mathcal{H}} \{ (Ff, f) + (Gg, g) : \ h = f + g \} \ ,$$

see [2], [19], [26]. One can establish for $F : G$ the following equivalent definition

$$F : G = s - \lim_{\varepsilon\downarrow 0} F\,(F + G + \varepsilon I)^{-1}\,G \ ,$$

see [3], [35]. Then for *positive definite* bounded self-adjoint operators F and G we obtain

$$F : G = (F^{-1} + G^{-1})^{-1} \ .$$

Since $F \le F + G$ and $G \le F + G$, one gets

$$F = (F + G)^{1/2} M (F + G)^{1/2}, \quad G = (F + G)^{1/2}(I - M)(F + G)^{1/2} \tag{2.4}$$

for some non-negative contraction M on $\mathcal{H}$ with $\operatorname{ran} M \subset \overline{\operatorname{ran}}\,(F + G)$. This yields yet another description of the parallel sum $F : G$.

Lemma 2.5. [5] *Suppose $F, G \in \mathbf{B}^+(\mathcal{H})$ and let M be as in* (2.4)*. Then*

$$F : G = (F + G)^{1/2}(M - M^2)(F + G)^{1/2}.$$

Using (2.1) and (2.4) one obtains the equalities

$$\operatorname{ran} F^{1/2} = (F + G)^{1/2}\operatorname{ran} M^{1/2}, \ \operatorname{ran} G^{1/2} = (F + G)^{1/2}\operatorname{ran}\,(I - M)^{1/2}.$$

Since

$$\operatorname{ran} M^{1/2} \cap \operatorname{ran}\,(I - M)^{1/2} = \operatorname{ran}\,(M - M^2)^{1/2},$$

the next proposition is an immediate consequence of Lemma 2.5, cf. [19], [35].

Proposition 2.6. *1)* $\operatorname{ran}\,(F : G)^{1/2} = \operatorname{ran} F^{1/2} \cap \operatorname{ran} G^{1/2}$.

2) The following statements are equivalent:

(i) $F : G = 0$*;*
(ii) *the operator M in* (2.4) *is an orthogonal projection in* $\overline{\operatorname{ran}}\,(F + G)$*;*
(iii) $\operatorname{ran} F^{1/2} \cap \operatorname{ran} G^{1/2} = \{0\}$.

2.5. **The Kreĭn shorted operator.** For a given non-negative bounded operator B on the Hilbert space $\mathcal{H}$ and for any subspace $\mathcal{K} \subset \mathcal{H}$ M.G. Kreĭn defined in [24] the operator

$$B_{\mathcal{K}} := \max\{\, Z \in \mathbf{B}(\mathcal{H}) : \ 0 \le Z \le B, \, \operatorname{ran} Z \subseteq \mathcal{K}\,\}.$$

Then equivalent definition of $B_{\mathcal{K}}$ has the following quadratic-form expression:

$$(B_{\mathcal{K}} f, f) := \inf_{\varphi \in \mathcal{K}^{\perp}} \{(B(f+\varphi), f+\varphi)\}, \quad f \in \mathcal{H}. \tag{2.5}$$

Here $\mathcal{K}^{\perp} := \mathcal{H} \ominus \mathcal{K}$. The operator $B_{\mathcal{K}}$ is called the *shorted operator* of B, see [1, 3]. Let the subspace $\Omega_{\mathcal{K}}$ be defined by

$$\Omega_{\mathcal{K}} := \{\, f \in \overline{\operatorname{ran}}\, B : \ B^{1/2} f \in \mathcal{K}\,\} = \overline{\operatorname{ran}}\, B \ominus B^{1/2}\mathcal{K}^{\perp}.$$

Then the shorted operator $B_{\mathcal{K}}$ gets the form $B_{\mathcal{K}} = B^{1/2} P_{\Omega_{\mathcal{K}}} B^{1/2}$ and

$$\operatorname{ran} B_{\mathcal{K}}^{1/2} = \mathcal{K} \cap \operatorname{ran} B^{1/2} \ ,$$

see [24]. In particular, this implies the equivalence:

$$B_{\mathcal{K}} = 0 \iff \mathcal{K} \cap \operatorname{ran} B^{1/2} = \{0\} \ . \tag{2.6}$$

Note that with respect to orthogonal decomposition $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}^{\perp}$ a bounded self-adjoint operator B has the block-matrix form:

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{12}^{*} & B_{22} \end{bmatrix} : \begin{matrix} \mathcal{K} \\ \oplus \\ \mathcal{K}^{\perp} \end{matrix} \to \begin{matrix} \mathcal{K} \\ \oplus \\ \mathcal{K}^{\perp} \end{matrix} \ ,$$

where $B_{11} \in \mathbf{B}(\mathcal{K})$, $B_{22} \in \mathbf{B}(\mathcal{K}^{\perp})$, $B_{12} \in \mathbf{B}(\mathcal{K}^{\perp}, \mathcal{K})$. It is well-known (see e.g.) Recall that the operator B is non-negative if and only if [25]

$$B_{22} \ge 0, \ \operatorname{ran} B_{12}^{*} \subset \operatorname{ran} B_{22}^{1/2}, \ \ B_{11} \ge \left(B_{22}^{[-1/2]} B_{12}^{*}\right)^{*} \left(B_{22}^{[-1/2]} B_{12}^{*}\right) \ . \tag{2.7}$$

Here $B_{22}^{[-1/2]} := (B_{22}^{1/2} \upharpoonright \overline{\operatorname{ran}}\, B_{22})^{-1}$. Then operator $B_{\mathcal{K}}$ is given by the block matrix

$$B_{\mathcal{K}} = \begin{bmatrix} B_{11} - \left(B_{22}^{[-1/2]} B_{12}^{*}\right)^{*} \left(B_{22}^{[-1/2]} B_{12}^{*}\right) & 0 \\ 0 & 0 \end{bmatrix}. \tag{2.8}$$

Conditions (2.7) imply that the block operator matrix

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{12}^{*} & B_{22} \end{bmatrix} : \begin{matrix} \mathcal{K} \\ \oplus \\ \mathcal{K}^{\perp} \end{matrix} \to \begin{matrix} \mathcal{K} \\ \oplus \\ \mathcal{K}^{\perp} \end{matrix}$$

is non-negative if and only if it takes the form [38]

$$B = \begin{bmatrix} B_{11} & B_{11}^{1/2} \Gamma B_{22}^{1/2} \\ B_{22}^{1/2} \Gamma^{*} B_{11}^{1/2} & B_{22} \end{bmatrix} \tag{2.9}$$

where $\Gamma : \overline{\operatorname{ran}}\, B_{22} \to \overline{\operatorname{ran}}\, B_{11}$ is a contraction. Then from (2.8) it follows that

$$B_{\mathcal{K}} = \begin{bmatrix} B_{11}^{1/2}(I - \Gamma^{*}\Gamma) B_{11}^{1/2} & 0 \\ 0 & 0 \end{bmatrix}, \ B_{\mathcal{K}^{\perp}} = \begin{bmatrix} 0 & 0 \\ 0 & B_{22}^{1/2}(I - \Gamma\Gamma^{*}) B_{22}^{1/2} \end{bmatrix}. \tag{2.10}$$

2.6. **Friedrichs and Kreĭn self-adjoint extensions.** Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{A}$ be a densely defined closed, symmetric, and non-negative operator. Denote by $\mathcal{A}^*$ the adjoint to $\mathcal{A}$. Recall that the operator $\mathcal{A}$ admits at least one non-negative self-adjoint extension $\mathcal{A}_{\rm F}$ (called the *Friedrichs*, or *"hard"* extension [24]), which is defined as follows. Denote by $\mathfrak{a}[\cdot,\cdot]$ the closure of corresponding to $\mathcal{A}$ sesquilinear form

$$\mathfrak{a}[f,g] = (\mathcal{A}f,g),\ f,g \in \mathrm{dom}\,(\mathcal{A})\ ,$$

and let $\mathcal{D}[\mathfrak{a}]$ be domain of this closure. According to the *first representation theorem* [20] there exists a unique non-negative self-adjoint operator $\mathcal{A}_F$ associated with $\mathfrak{a}[\cdot,\cdot]$, i.e.,

$$(\mathcal{A}_{\rm F}h,\psi) = \mathfrak{a}[h,\psi],\ \psi \in \mathcal{D}[\mathfrak{a}],\ h \in \mathrm{dom}\,\mathcal{A}_{\rm F}\ .$$

One clearly gets that $\mathcal{A} \subset \mathcal{A}_{\rm F} \subset \mathcal{A}^*$ and that $\mathrm{dom}\,\mathcal{A}_{\rm F} = \mathcal{D}[\mathfrak{a}] \cap \mathrm{dom}\,\mathcal{A}^*$. Moreover, by the *second representation theorem* [20] the following equalities

$$\mathcal{D}[\mathfrak{a}] = \mathrm{dom}\,\mathcal{A}_{\rm F}^{1/2} \quad \text{and} \quad \mathfrak{a}[\phi,\psi] = (\mathcal{A}_{\rm F}^{1/2}\phi, \mathcal{A}_{\rm F}^{1/2}\psi),\quad \phi,\psi \in \mathcal{D}[\mathfrak{a}]\ ,$$

also hold.

In [24] M.G. Kreĭn discovered one more non-negative self-adjoint extension $\mathcal{A}_{\rm K}$ of $\mathcal{A}$. It has the extremal property to be a *minimal*, whereas the Friedrichs extension $\mathcal{A}_{\rm F}$ is the *maximal* (in the sense of the corresponding associated closed quadratic forms) among *all* other non-negative self-adjoint extensions $\mathcal{C}$ of $\mathcal{A}$: $\mathcal{A}_{\rm K} \leq \mathcal{C} \leq \mathcal{A}_{\rm F}$. These inequalities are equivalent to inequalities for resolvents :

$$(\mathcal{A}_{\rm F} + aI)^{-1} \leq (\mathcal{C} + aI)^{-1} \leq (\mathcal{A}_{\rm K} + aI)^{-1},\ a > 0\ ,$$

see [20], [24]. The extension $\mathcal{A}_{\rm K}$ is called the *Kreĭn extension* of $\mathcal{A}$. If $\mathcal{A}$ is a positive-definite symmetric operator, then the subspace $\ker \mathcal{A}^*$ is nontrivial and one gets:

$$\mathrm{dom}\,\mathcal{A}_{\rm K} = \mathrm{dom}\,\mathcal{A} \dot{+} \ker \mathcal{A}^*\ ,$$

see [24], whereas

$$\mathrm{dom}\,\mathcal{A}_{\rm F} = \mathrm{dom}\,\mathcal{A} \dot{+} (\mathcal{A}_{\rm F})^{-1} \ker \mathcal{A}^*\ .$$

Let L_1 and L_2 be closed linear operators defined in a Hilbert space $\mathcal{H}$, taking values in a Hilbert space $\mathfrak{H}$, such that L_2 is extension of L_1:

$$L_1 \subset L_2. \tag{2.11}$$

Then operators $L_1^*L_1$ and $L_2^*L_2$ are self-adjoint and non-negative. Since $L_2^* \subset L_1^*$, the following relations are valid:

$$\mathrm{dom}\,(L_2^*L_1) = \mathrm{dom}\,(L_1^*L_1) \cap \mathrm{dom}\,(L_2^*L_2) = \mathrm{dom}\,L_1 \cap \mathrm{dom}\,(L_2^*L_2).$$

Suppose that

$$\mathrm{dom}\,(L_1^*L_1) \cap \mathrm{dom}\,(L_2^*L_2) \neq \{0\}\ . \tag{2.12}$$

Then operator $\mathcal{A}$ defined as follows:

$$\mathcal{A}f := L_2^*L_1 f,\ f \in \mathrm{dom}\,\mathcal{A}, \quad \text{for}\ \ \mathrm{dom}\,\mathcal{A} := \mathrm{dom}\,(L_2^*L_1)\ , \tag{2.13}$$

is closed and symmetric. Since $(\mathcal{A}f,f) = ||L_1 f||^2 \geq 0$ for all $f \in \mathrm{dom}\,\mathcal{A}$, the operator $\mathcal{A}$ is non-negative. This kind of operators $\mathcal{A}$ we call the *operators in divergence form* [7]. The next assertions are established in [7].

Theorem 2.7. [7, Theorem 3.1]. *Let $L_1, L_2 : \mathcal{H} \to \mathfrak{H}$ be closed and densely defined operators, satisfying condition* (2.11)*. If the operator $\mathcal{A} = L_2^* L_1$ is densely defined and its adjoint is given by*

$$\mathcal{A}^* = L_1^* L_2,$$

then

(1) *the Friedrichs extension of $\mathcal{A}$ is given by the operator $L_1^* L_1$, i.e.,*

$$\mathrm{dom}\,\mathcal{A}_{\mathrm{F}} = \{f \in \mathrm{dom}\,L_1 : L_1 f \in \mathrm{dom}\,L_1^*\},\ \mathcal{A}_{\mathrm{F}} f = L_1^* L_1 f,\ f \in \mathrm{dom}\,\mathcal{A}_{\mathrm{F}},$$

(2)

$$\mathrm{dom}\,\mathcal{A}_{\mathrm{F}}^{1/2} = \mathrm{dom}\,L_1,\ (\mathcal{A}_{\mathrm{F}}^{1/2} u, \mathcal{A}_{\mathrm{F}}^{1/2} v) = (L_1 u, L_1 v),\ u, v \in \mathrm{dom}\,L_1,$$

(3) *the Kreĭn extension of $\mathcal{A}$ is the operator $\mathcal{A}_{\mathrm{K}} = L_2^* P_{\overline{\mathrm{ran}}\,L_1} L_2$, i.e.,*

$$\begin{aligned}&\mathrm{dom}\,\mathcal{A}_{\mathrm{K}} = \{f \in \mathrm{dom}\,L_2 : P_{\overline{\mathrm{ran}}\,L_1} L_2 f \in \mathrm{dom}\,L_2^*\},\\ &\mathcal{A}_{\mathrm{K}} f = L_2^* P_{\overline{\mathrm{ran}}\,L_1} L_2 f,\ f \in \mathrm{dom}\,\mathcal{A}_{\mathrm{K}},\end{aligned}$$

and

$$\mathrm{dom}\,\mathcal{A}_{\mathrm{K}}^{1/2} = \mathrm{dom}\,L_2,\ (\mathcal{A}_{\mathrm{K}}^{1/2} u, \mathcal{A}_{\mathrm{K}}^{1/2} v) = (P_{\overline{\mathrm{ran}}\,L_1} L_2 u, P_{\overline{\mathrm{ran}}\,L_1} L_2 v),\ u, v \in \mathrm{dom}\,L_2,$$

(4) *the Friedrichs and the Kreĭn extensions of $\mathcal{A}$ are transversal, i.e., one gets for their domains:*

$$\mathrm{dom}\,\mathcal{A}_{\mathrm{F}} + \mathrm{dom}\,\mathcal{A}_{\mathrm{K}} = \mathrm{dom}\,\mathcal{A}^*.$$

3. Main results

This section collects our main results. They are based on some new ideas and our lines reasoning improve the results in [39], [36], [11]. We give new proofs and generalise the Van Daele–Schmüdgen Theorems 1.1,1.2,1.4,1.7 and the Brasche–Neidhardt assertion [11].

Our observations also lead to certain new applications, see Section 3.3.

3.1. **Trivial intersections of operator ranges with subspaces.**

We start this section by a useful refinement of the von Neumann Theorem 2.2, which we reformulated in Theorem 2.3 in terms of ranges.

A bounded linear operator J on a Hilbert space $\mathcal{H}$ is self-adjoint and unitary operator if and only if one has:

$$J = J^* = J^{-1}\ .$$

Such operator is often called *fundamental symmetry*, or *signature operator* [10]. Note that J is a fundamental symmetry operator if and only if

$$J = 2P - I,$$

where P is an orthogonal projection in $\mathcal{H}$.

Proposition 3.1. *Let $\mathcal{R}$ be a non-closed linear manifold in a Hilbert space $\mathcal{H}$. Then the following assertions are equivalent:*

(i) *There exists in $\mathcal{H}$ an orthogonal projection P such that*

$$\mathrm{ran}\,P \cap \mathcal{R} = \{0\} \quad \text{and} \quad \mathrm{ran}\,(I-P) \cap \mathcal{R} = \{0\}\ .$$

(ii) *There exists in $\mathcal{H}$ a fundamental symmetry J such that*

$$J\mathcal{R} \cap \mathcal{R} = \{0\}\ .$$

Proof. (i) $\Rightarrow$ (ii). Set $J := 2P - I$. Let $f \in \mathcal{R}$ and suppose $Jf \in \mathcal{R}$. Then $2Pf \in \mathcal{R}$. But $\operatorname{ran} P \cap \mathcal{R} = \{0\}$. Hence $f \in \operatorname{ran}(I-P)$. Since $\operatorname{ran}(I-P) \cap \mathcal{R} = \{0\}$, we obtain $f = 0$, i.e., the statement (ii) holds.

(ii) $\Rightarrow$ (i). Let $P := (I+J)/2$. Then P is orthogonal projection in $\mathcal{H}$. Suppose that $f \in (\operatorname{ran} P \cap \mathcal{R})$. Then $Jf = Pf = f \in \mathcal{R}$ and by virtue of (ii) one obtains $f = 0$. A similar argument is valid for $f \in (\operatorname{ran}(I-P) \cap \mathcal{R} = \{0\})$. □

Proposition 3.2. *Let $\mathcal{H}$ be a Hilbert space. Let $A \in \mathbf{B}_0^+(\mathcal{H})$ and $\operatorname{ran} A \neq \mathcal{H}$.*

(1) *Let $\mathfrak{M}$ be a subspace in $\mathcal{H}$ and $P_{\mathfrak{M}}$ be orthogonal projection on $\mathfrak{M}$. We define the operator $A_1 := A^{1/2} P_{\mathfrak{M}} A^{1/2}$.*

(a) *Then one gets*

$$\mathfrak{M}^\perp \cap \operatorname{ran} A^{1/2} = \{0\} \iff \ker A_1 = \{0\} \iff \overline{A^{1/2}\mathfrak{M}} = \mathcal{H},$$
$$\mathfrak{M} \cap \operatorname{ran} A^{1/2} = \{0\} \iff \operatorname{ran} A_1^{1/2} \cap \operatorname{ran} A = \{0\}.$$

Hence, the following statements are equivalent:

(i) $\operatorname{ran} A_1^{1/2} \cap \operatorname{ran} A = \{0\}$ *and* $\ker A_1 = \{0\}$,

(ii) $\mathfrak{M} \cap \operatorname{ran} A^{1/2} = \mathfrak{M}^\perp \cap \operatorname{ran} A^{1/2} = \{0\}$,

(iii) *the linear manifolds $A^{1/2}\mathfrak{M}$ and $A^{1/2}\mathfrak{M}^\perp$ are dense in $\mathcal{H}$.*

(b) *If* $\ker A_1 = \{0\}$, *then*

$$||A_1^{-1/2}h|| = ||A^{-1/2}h|| \text{ for all } h \in \operatorname{ran} A_1^{1/2}. \tag{3.1}$$

(2) *If $A, A_1 \in \mathbf{B}^+(\mathcal{H})$, $\operatorname{ran} A_1^{1/2} \subset \operatorname{ran} A^{1/2}$ and if* (3.1) *holds true, then $A_1 = A^{1/2} P A^{1/2}$, where P is an orthogonal projection in $\mathcal{H}$.*

Proof. (1) By definition of A_1 and by the Douglas Theorem 2.1 we have $\operatorname{ran} A_1^{1/2} = A^{1/2}\mathfrak{M}$. It follows then that

$$\operatorname{ran} A_1^{1/2} \cap \operatorname{ran} A = \{0\} \iff \mathfrak{M} \cap \operatorname{ran} A^{1/2} = \{0\}.$$

It is also clear that

$$\ker A_1 = \{0\} \iff \overline{\operatorname{ran}}\, A_1 = \mathcal{H} \iff \mathfrak{M}^\perp \cap \operatorname{ran} A^{1/2} = \{0\}.$$

The equality: $||A_1^{1/2} f||^2 = ||P_{\mathfrak{M}} A^{1/2} f||^2$ for all $f \in \mathcal{H}$, implies that there is an isometry $V : \mathfrak{M} \to \mathcal{H}$, $\operatorname{ran} V = \mathcal{H}$ such that $A_1^{1/2} h = V P_{\mathfrak{M}} A^{1/2} h$, $h \in \mathcal{H}$. Hence

$$A_1^{1/2} = A^{1/2} V^*, \;\; A^{-1/2} h = V^* A_1^{-1/2} h, \;\; h \in \operatorname{ran} A_1^{1/2},$$

where $V^* : \mathcal{H} \to \mathfrak{M}$, $\operatorname{ran} V^* = \mathfrak{M}$ and V^* is isometry.

For the proof of the statement (2) we refer to [8]. □

Proposition 3.3. *Let $A \in \mathbf{B}_0^+(\mathcal{H})$ and $\operatorname{ran} A \neq \mathcal{H}$. Let P_1 and P_2 be two orthogonal projections in $\mathcal{H}$ such that*

$$\operatorname{ran} P_k \cap \operatorname{ran} A^{1/2} = \operatorname{ran}(I - P_k) \cap \operatorname{ran} A^{1/2} = \{0\}, \;\; k = 1, 2 \tag{3.2}$$

If we define

$$A_1 := A^{1/2} P_1 A^{1/2}, \;\; A_2 := A_1^{1/2} P_2 A_1^{1/2},$$

then

$$A_2 = A^{1/2} P_{12} A^{1/2},$$

where P_{12} is an orthogonal projection such that

$$\operatorname{ran} P_{12} \cap \operatorname{ran} A^{1/2} = \operatorname{ran}(I - P_{12}) \cap \operatorname{ran} A^{1/2} = \{0\}. \tag{3.3}$$

Proof. We have $A_1^{1/2} = V_1 P_1 A^{1/2}$, $A_2^{1/2} = V_2 P_2 A_1^{1/2}$, where $V_1 : \operatorname{ran} P_1 \to \mathcal{H}$, $V_2 : \operatorname{ran} P_2 \to \mathcal{H}$ are isometries. Then

$$A_1^{1/2} = A^{1/2} V_1^*,\ A_2^{1/2} = A_1^{1/2} V_2^*,\ V_k^* : \mathcal{H} \to \operatorname{ran} P_k,\ k = 1, 2.$$

It follows that

$$A_2^{1/2} = A^{1/2} V_2^* V_1^*.$$

The operator $V := V_2^* V_1^*$ is isometry,

$$\operatorname{ran} V = V_2^* \{\operatorname{ran} V_1^*\} = V_2^* \{\operatorname{ran} P_1\} \subset \operatorname{ran} P_2.$$

Let $P_{12} := P_{\operatorname{ran} V}$ be orthogonal projection on $\operatorname{ran} V$. Then

$$A_2 = A^{1/2} V V^* A^{1/2} = A^{1/2} P_{12} A^{1/2}.$$

Using (3.2) and Proposition 3.2, we obtain that $\ker A_2 = \{0\}$, $\operatorname{ran} A_2^{1/2} \cap \operatorname{ran} A_1 = \{0\}$, and $\operatorname{ran} A_1^{1/2} \cap \operatorname{ran} A = \{0\}$. Due to inclusion $\operatorname{ran} A_2^{1/2} \subset \operatorname{ran} A_1^{1/2}$, one gets $\operatorname{ran} A_2^{1/2} \cap \operatorname{ran} A = \{0\}$. Then application of Proposition 3.2 leads to equalities (3.3). □

Proposition 3.4. *Let $A \in \mathbf{B}_0^+(\mathcal{H})$, $\operatorname{ran} A \neq \mathcal{H}$ and let P_1 and P_2 be two orthogonal projections in $\mathcal{H}$ such that $P_1 \le P_2$ and (3.2) holds. Define*

$$A_1 := A^{1/2} P_1 A^{1/2},\ A_2 := A^{1/2} P_2 A^{1/2}.$$

Then

$$A_1 = A_2^{1/2} P A_2^{1/2},$$

where P is an orthogonal projection such that

$$\operatorname{ran}(I - P) \cap \operatorname{ran} A_2^{1/2} = \{0\}.$$

Proof. It is clear that

$$A_1^{1/2} = V_1 P_1 A^{1/2},\ A_2^{1/2} = V_2 P_2 A^{1/2},$$

where the operator $V_k : \operatorname{ran} P_k \to \mathcal{H}$ is isometry, $k = 1, 2$. Then

$$A_1^{1/2} = A_2^{1/2} V_2 V_1^*$$

and $V := V_2 V_1^*$ is isometry with $\operatorname{ran} V = V_2\{\operatorname{ran} P_1\}$. Hence

$$A_1 = A_2^{1/2} P A_2^{1/2},$$

where $P := P_{\operatorname{ran} V}$ is orthogonal projection on $\operatorname{ran} V$. Since $\ker A_1 = \{0\}$, we obtain the last statement, $\operatorname{ran}(I - P) \cap \operatorname{ran} A_2^{1/2} = \{0\}$, of the theorem. □

Proposition 3.5. *Let $\mathcal{H}$ be a Hilbert space.*

1) Suppose that

$$\begin{array}{l} F,\ G \in \mathbf{B}^+(\mathcal{H}),\ \ker F = \ker G = \{0\}, \\ F : G = 0 (\iff \operatorname{ran} F^{1/2} \cap \operatorname{ran} G^{1/2} = \{0\}). \end{array} \tag{3.4}$$

Then there is a subspace $\mathfrak{M}$ in $\mathcal{H}$ satisfying

$$\mathfrak{M} \cap \operatorname{ran}(F + G)^{1/2} = \mathfrak{M}^\perp \cap \operatorname{ran}(F + G)^{1/2} = \{0\}. \tag{3.5}$$

2) If a subspace $\mathfrak{M}$ in $\mathcal{H}$ is such that $\dim \mathfrak{M} = \dim \mathfrak{M}^\perp = \infty$, *then there is a pair of linear operators F and G satisfying* (3.4) *and* (3.5) *.*

Proof. By virtue of Proposition 2.6 and equalities (2.4) we have

$$F = (F+G)^{1/2} P (F+G)^{1/2}, \quad G = (F+G)^{1/2}(I-P)(F+G)^{1/2},$$

where P is an orthogonal projection in $\mathcal{H}$. Put $\mathfrak{M} := \operatorname{ran} P$. Then, since $\ker F = \ker G = \{0\}$, we obtain (3.5).

Conversely, suppose that $\mathfrak{M}$ is a subspace in $\mathcal{H}$ such that $\dim \mathfrak{M} = \dim \mathfrak{M}^\perp = \infty$. Then by [8] one can find operator $X \in \mathbf{B}(\mathcal{H})$, $X \geq 0$, such that $\ker X = \{0\}$ and

$$\mathfrak{M} \cap \operatorname{ran} X^{1/2} = \{0\},\ \mathfrak{M}^\perp \cap \operatorname{ran} X^{1/2} = \{0\}. \tag{3.6}$$

Now we follow the line of reasoning close to constructions in [8, Section 5]. To this end note that by Theorems 2.2 and 2.3 one can find in the subspace $\mathfrak{M}$ two non-negative self-adjoint operators W and V from $\mathbf{B}(\mathfrak{M})$ possessing the properties

$$\begin{array}{l}\overline{\operatorname{ran}}\, W = \overline{\operatorname{ran}}\, V = \mathfrak{M},\ \operatorname{ran} W \cap \operatorname{ran} V = \{0\},\\ 0 \leq W \leq I_{\mathfrak{M}},\ \ker W = \{0\},\ 0 \leq V \leq I_{\mathfrak{M}},\ \ker V = \{0\}.\end{array}$$

Let us replace V by operator $U = V\Phi$, where Φ is a unitary operator from $\mathfrak{M}^\perp$ onto $\mathfrak{M}$ and taking into account $\mathcal{H} = \mathfrak{M} \oplus \mathfrak{M}^\perp$, define

$$X := \begin{bmatrix} W^2 & WU \\ U^*W & U^*U \end{bmatrix}.$$

Let us show that

$$\ker X = \{0\},\ X_{\mathfrak{M}} = 0,\ X_{\mathfrak{M}^\perp} = 0.$$

Set $f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$, where $f_1 \in \mathfrak{M},\ f_2 \in \mathfrak{M}^\perp$. Then

$$(Xf, f) = ||Wf_1 + Uf_2||^2. \tag{3.7}$$

It follows that

$$Xf = 0 \iff Wf_1 + Uf_2 = 0.$$

Since $\operatorname{ran} W \cap \operatorname{ran} U = \{0\}$, $\ker W = \{0\}$, $\ker U = \{0\}$, we get $f_1 = 0,\ f_2 = 0$. From (3.7) and relations $\overline{\operatorname{ran}}\, W = \overline{\operatorname{ran}}\, U = \mathfrak{M}$ we get the equalities

$$\inf_{\varphi \in \mathfrak{M}^\perp} (X(f-\varphi), f-\varphi) = 0,\ \inf_{\psi \in \mathfrak{M}} (X(f-\psi), f-\psi) = 0.$$

Equality (2.5) now implies $X_{\mathfrak{M}} = 0$ and $X_{\mathfrak{M}^\perp} = 0$. Applying (2.6) we obtain (3.6).

Now set

$$F = X^{1/2} P_{\mathfrak{M}} X^{1/2},\ G = X^{1/2}(I - P_{\mathfrak{M}})X^{1/2}.$$

Then by construction $\ker F = \ker G = \{0\}$, $\operatorname{ran} F^{1/2} = X^{1/2}\mathfrak{M}$, $\operatorname{ran} G^{1/2} = X^{1/2}\mathfrak{M}^\perp$. Hence

$$\operatorname{ran} F^{1/2} \cap \operatorname{ran} G^{1/2} = \{0\},$$

therefore by Proposition 2.6 one obtains $F : G = 0$. Since $F + G = X$, this proves the statement 2). Hence, the proof of the proposition is completed. ☐

Corollary 3.6. *Let $X\in\mathbf{B}_0^+(\mathcal{H})$, and a subspace $\mathfrak{M}\subset\mathcal{H}$. Then*
$$\mathfrak{M}\cap\operatorname{ran}X^{1/2}=\mathfrak{M}^\perp\cap\operatorname{ran}X^{1/2}=\{0\}$$
if and only if the operator X with respect to decomposition $\mathcal{H}=\mathfrak{M}\oplus\mathfrak{M}^\perp$ takes the form
$$X=\begin{bmatrix}W^2 & WU\\ U^*W & U^*U\end{bmatrix},\tag{3.8}$$
where
$$\begin{array}{l}W\in\mathbf{B}_0^+(\mathfrak{M}),\; U\in\mathbf{B}(\mathfrak{M}^\perp,\mathfrak{M}),\;\ker U=\{0\},\;\ker U^*=\{0\},\\ \operatorname{ran}W\cap\operatorname{ran}U=\{0\}.\end{array}\tag{3.9}$$

Proof. If X is of the form (3.8) with conditions (3.9) then due to (3.7), (2.6), (2.8) we get that
$$\mathfrak{M}\cap\operatorname{ran}X^{1/2}=\mathfrak{M}^\perp\cap\operatorname{ran}X^{1/2}=\{0\},\;\ker X=\{0\}.$$
Conversely, suppose $X\in\mathbf{B}_0^+(\mathcal{H})$ and $\operatorname{ran}X^{1/2}\cap\mathfrak{M}=\operatorname{ran}X^{1/2}\cap\mathfrak{M}^\perp=\{0\}$. From (2.6) and (2.9), (2.10) we get that with respect to orthogonal decomposition $\mathcal{H}=\mathfrak{M}\oplus\mathfrak{M}^\perp$ the operator X takes the form
$$X=\begin{bmatrix}X_{11} & X_{11}^{1/2}\Gamma X_{22}^{1/2}\\ X_{22}^{1/2}\Gamma^*X_{11}^{1/2} & X_{22}\end{bmatrix},$$
where $\ker X_{11}=\{0\}$, $\ker X_{22}=\{0\}$, and Γ is unitary map of $\mathfrak{M}^\perp$ onto $\mathfrak{M}$. Denote $W=X_{11}^{1/2}$, $U=\Gamma X_{22}^{1/2}$. Then X is of the form (3.8). Moreover, $\operatorname{ran}W\cap\operatorname{ran}U=\{0\}$ due to $\ker X=\{0\}$. Therefore, conditions (3.9) are satisfied. □

Let $U=V\Phi$ be the polar decomposition of $U\in\mathbf{B}(\mathfrak{M}^\perp,\mathfrak{M})$, where $V=(UU^*)^{1/2}$ and Φ is unitary operator acting from $\mathfrak{M}^\perp$ onto $\mathfrak{M}$. Then X in (3.8) takes the form
$$X=\begin{bmatrix}W^2 & WV\Phi\\ \Phi^*VW & \Phi^*V^2\Phi\end{bmatrix}:\begin{array}{c}\mathfrak{M}\\ \oplus\\ \mathfrak{M}^\perp\end{array}\to\begin{array}{c}\mathfrak{M}\\ \oplus\\ \mathfrak{M}^\perp\end{array},\tag{3.10}$$
and
$$\begin{array}{l}W\in\mathbf{B}_0^+(\mathfrak{M}),\; V\in\mathbf{B}_0^+(\mathfrak{M}),\\ \operatorname{ran}W\cap\operatorname{ran}V=\{0\}.\end{array}\tag{3.11}$$

Let us formulate a *general* criterion: The operator $X\in\mathbf{B}_0^+(\mathcal{H})$, having the block-operator matrix form
$$X=\begin{bmatrix}X_{11} & X_{12}\\ X_{12}^* & X_{22}\end{bmatrix}:\begin{array}{c}\mathfrak{M}\\ \oplus\\ \mathfrak{M}^\perp\end{array}\to\begin{array}{c}\mathfrak{M}\\ \oplus\\ \mathfrak{M}^\perp\end{array},$$
possess the property
$$\mathfrak{M}\cap\operatorname{ran}X^{1/2}=\mathfrak{M}^\perp\cap\operatorname{ran}X^{1/2}=\{0\}\;,$$
if and only if
$$\ker X_{11}=\{0\},\;\ker X_{22}=\{0\},\;\operatorname{ran}X_{12}\cap\operatorname{ran}X_{11}=\{0\}.$$

Now we are in position to formulate the first of our main results of this section. It concerns subspaces that have trivial intersections with the operator range $\operatorname{ran}A^{1/2}$.

Theorem 3.7. *Let $A \in \mathbf{B}_0^+(\mathcal{H})$ and $\operatorname{ran} A \neq \mathcal{H}$. Then there is a continuum set of subspaces $\mathfrak{M} \subset \mathcal{H}$ such that*

$$\mathfrak{M} \cap \operatorname{ran} A^{1/2} = \mathfrak{M}^\perp \cap \operatorname{ran} A^{1/2} = \{0\}. \tag{3.12}$$

Proof. By Theorem 2.3 there exists $B \in \mathbf{B}_0^+(\mathcal{H})$ such that $\operatorname{ran} B^{1/2} \cap \operatorname{ran} A^{1/2} = \{0\}$ (take for instance $B = UAU^*$, where U is unitary in $\mathcal{H}$ and satisfies (2.3)). Then the parallel sum $A : B = 0$, and hence by Theorem 3.5 there is a subspace $\mathfrak{M}$ such that

$$\mathfrak{M} \cap \operatorname{ran}(A+B)^{1/2} = \mathfrak{M}^\perp \cap \operatorname{ran}(A+B)^{1/2} = \{0\}.$$

Since $\operatorname{ran}(A+B)^{1/2} = \operatorname{ran} A^{1/2} + \operatorname{ran} B^{1/2}$, we get (3.12). Notice that

$$A = (A+B)^{1/2} P_{\mathfrak{M}} (A+B)^{1/2},\ B = (A+B)^{1/2} P_{\mathfrak{M}^\perp} (A+B)^{1/2}.$$

Let $M \in \mathbf{B}^+(\mathcal{H})$. Then

$$\operatorname{ran}(B^{1/2} M B^{1/2})^{1/2} = B^{1/2} \operatorname{ran} M^{1/2} \subseteq \operatorname{ran} B^{1/2}.$$

Since $\operatorname{ran} B^{1/2} \cap \operatorname{ran} A^{1/2} = \{0\}$, then

$$\operatorname{ran}(B^{1/2} M B^{1/2})^{1/2} \cap \operatorname{ran} A^{1/2} = \{0\}\ .$$

Hence

$$\begin{aligned} &A = (A + B^{1/2} M B^{1/2})^{1/2} P(M) (A + B^{1/2} M B^{1/2})^{1/2},\\ &B^{1/2} M B^{1/2} = (A + B^{1/2} M B^{1/2})^{1/2} (I - P(M)) (A + B^{1/2} M B^{1/2})^{1/2}, \end{aligned}$$

where $P(M)$ is orthogonal projection in $\mathcal{H}$. In particular

$$A = (A + xB)^{1/2} P(x) (A + xB)^{1/2},\ xB = (A + xB)^{1/2} (I - P(x)) (A + xB)^{1/2}\ , \tag{3.13}$$

for proportional to identity operators $M(x) = xI$ with positive parameter x. Here we put $P(x) := P(M(x))$. Then

$$\operatorname{ran} P(x) \cap \operatorname{ran} A^{1/2} = \operatorname{ran}(I - P(x)) \cap \operatorname{ran} A^{1/2} = \{0\}.$$

We show first that if $x, y > 0$ and $x \neq y$, then $P(x) \neq P(y)$. Notice that

$$\operatorname{ran}(A + xB)^{1/2} = \operatorname{ran} A^{1/2} \dot{+} \operatorname{ran} B^{1/2},\ x > 0. \tag{3.14}$$

Suppose that $x < y$. Then

$$A + xB \le A + yB.$$

Hence, by the Douglas Theorem 2.1 one obtains

$$(A + xB)^{1/2} = Z_{x,y} (A + yB)^{1/2}, \tag{3.15}$$

where $Z_{x,y} \in \mathbf{B}(\mathcal{H})$ is a contraction. Note that

$$Z_{x,y}^* = (A + yB)^{-1/2} (A + xB)^{1/2}.$$

Then equality (3.14) implies that the operators $Z_{x,y}^*$ as well as $Z_{x,y}$ are isomorphisms of $\mathcal{H}$. The first equality in (3.13) yields that

$$A = (A + yB)^{1/2} Z_{x,y}^* P(x) Z_{x,y} (A + yB)^{1/2}.$$

On the other hand

$$A = (A + yB)^{1/2} P(y) (A + yB)^{1/2}.$$

Thus

$$P(y) = Z_{x,y}^* P(x) Z_{x,y}. \tag{3.16}$$

Set $\mathfrak{M}_x = \operatorname{ran} P(x)$, $\mathfrak{M}_y = \operatorname{ran} P(y)$. From (3.16) we get that

(1) $Z_{x,y}$ maps $\mathfrak{M}_y^\perp$ into $\mathfrak{M}_x^\perp$,

(2) $Z_{x,y}$ maps $\mathfrak{M}_y$ into $\mathfrak{M}_x$ isometrically.

In fact $Z_{x,y}\mathfrak{M}_y = \mathfrak{M}_x$ and $Z_{x,y}\mathfrak{M}_y^\perp = \mathfrak{M}_x^\perp$, because $Z_{x,y}$ is isomorphism of $\mathcal{H}$. The equalities

$$\begin{aligned} xB &= (A+yB)^{1/2} Z_{x,y}^*(I-P(x))Z_{x,y}(A+yB)^{1/2},\\ yB &= (A+yB)^{1/2}(I-P(y))(A+yB)^{1/2} \end{aligned}$$

lead to

$$y\, Z_{x,y}^*(I-P(x))Z_{x,y} = x\,(I-P(y)),$$

and taking into account (3.16) we arrive at

$$(y-x)P(y) = yZ_{x,y}^*Z_{x,y} - xI.$$

Finally

$$y||Z_{x,y}h||^2 = x||h||^2,\ h\in\mathfrak{M}_y^\perp.$$

This equality means that the operator

$$\sqrt{yx^{-1}}\, Z_{x,y}$$

isometrically maps $\mathfrak{M}_y^\perp$ onto $\mathfrak{M}_x^\perp$. Now assume $P(x) = P(y)$, i.e., $\mathfrak{M}_x = \mathfrak{M}_y$. Denote this subspace by $\mathfrak{M}$. Then with respect to decomposition $\mathcal{H} = \mathfrak{M}\oplus\mathfrak{M}^\perp$ the operator $Z_{x,y}$ takes the matrix form

$$Z_{x,y} = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \sqrt{xy^{-1}}\,\Lambda_2 \end{bmatrix} : \begin{array}{c}\mathfrak{M}\\ \oplus \\ \mathfrak{M}^\perp\end{array} \to \begin{array}{c}\mathfrak{M}\\ \oplus \\ \mathfrak{M}^\perp\end{array},$$

where Λ_1 and Λ_2 are unitary operators in $\mathfrak{M}$ and $\mathfrak{M}^\perp$, respectively. Since

$$A = (A+yB)^{1/2}P_{\mathfrak{M}}(A+yB)^{1/2} = (A+xB)^{1/2}P_{\mathfrak{M}}(A+yB)^{1/2},$$

from (3.15) follows

$$A = Z_{x,y}(A+yB)^{1/2}P_{\mathfrak{M}}(A+yB)^{1/2}Z_{x,y}^*.$$

Therefore

$$A = Z_{x,y}^* A Z_{x,y}.$$

Due to the structure of $Z_{x,y}$, the latter equality implies

$$P_{\mathfrak{M}}^\perp A{\upharpoonright}\mathfrak{M}^\perp = Z_{x,y}^*(P_{\mathfrak{M}}^\perp A{\upharpoonright}\mathfrak{M}^\perp)Z_{x,y}{\upharpoonright}\mathfrak{M}^\perp.$$

Hence

$$\|P_{\mathfrak{M}}^\perp A{\upharpoonright}\mathfrak{M}^\perp\| = xy^{-1}\|\Lambda_2^{-1}(P_{\mathfrak{M}}^\perp A{\upharpoonright}\mathfrak{M}^\perp)\Lambda_2\|.$$

Because Λ_2 is unitary in $\mathfrak{M}^\perp$, we get

$$\|\Lambda_2^{-1}(P_{\mathfrak{M}}^\perp A{\upharpoonright}\mathfrak{M}^\perp)\Lambda_2\| = \|P_{\mathfrak{M}}^\perp A{\upharpoonright}\mathfrak{M}^\perp\|.$$

Thus,

$$\|P_{\mathfrak{M}}^\perp A{\upharpoonright}\mathfrak{M}^\perp\| = xy^{-1}\|P_{\mathfrak{M}}^\perp A{\upharpoonright}\mathfrak{M}^\perp\|.$$

Since $x\neq y$, this equality implies: $P_{\mathfrak{M}}^\perp A\upharpoonright\mathfrak{M}^\perp = 0$, i.e. the contradiction with $A\in\mathbf{B}_0^+(\mathcal{H})$. So, $P(x)\neq P(y)$ if $x\neq y$. □

It should be noted that the function $P(x)$ is *strongly* continuous at each point on $(0,+\infty)$. To prove this we define the following auxiliary operator-valued function

$$S_x := (A+xB)^{-1/2}A^{1/2}\ , \quad x>0\ .$$

Then

$$S_x^* h = A^{1/2}(A+xB)^{-1/2}h, \quad h\in \operatorname{ran} A^{1/2}\dot{+}\operatorname{ran} B^{1/2}\ ,$$

and from (3.13) one gets

$$S_x S_x^* = P(x). \tag{3.17}$$

Let $x_0>0$, then

$$\begin{aligned}\left(S_x^* - S_{x_0}^*\right)(A+x_0B)f = (x_0-x)A^{1/2}(A+xB)^{-1/2}Bf \\ + A^{1/2}\left((A+xB)^{1/2}-(A+x_0B)^{1/2}\right)f.\end{aligned}$$

Note that the Douglas Theorem 2.1 implies: $\sqrt{x}||(A+xB)^{-1/2}B^{1/2}||\le 1$. Therefore,

$$||(A+xB)^{-1/2}B^{1/2}||\le C$$

for all x in some neighborhood of the point x_0. Hence,

$$\lim_{x\to x_0}\left(S_x^* - S_{x_0}^*\right)(A+x_0B)f = 0$$

for all $f\in\mathcal{H}$. Since the linear manifold $\operatorname{ran}(A+x_0B)$ is dense in $\mathcal{H}$ and $||S_x^*||=1$, we obtain

$$\lim_{x\to x_0} S_x^* g = S_{x_0}^* g$$

for all $g\in\mathcal{H}$. From (3.17) it follows that $||S_x^* g|| = ||P(x)g||$, then

$$\lim_{x\to x_0}||P(x)g|| = ||P(x_0)g||,\ g\in\mathcal{H}. \tag{3.18}$$

On the other hand,

$$(P(x)g-P(x_0)g,f) = (S_xS_x^*g,f)-(S_{x_0}S_{x_0}^*g,f) = (S_x^*g,S_x^*f)-(S_{x_0}^*g,S_{x_0}^*f).$$

Hence, the function $P(x)$ is weakly continuous at x_0, which together with (3.18) implies that $P(x)$ is strongly continuous at x_0.

Proposition 3.8. *Let $\{U_t\}_{t\in\mathbb{R}}$ be a one-parameter unitary group such that*

$$U_t \operatorname{ran} A^{1/2}\cap U_s\operatorname{ran} A^{1/2} = \{0\},\ s\ne t,$$

see Corollary 2.4, then there is a one-parameter family of subspaces $\{\mathfrak{M}_t\}_{t\in\mathbb{R}\setminus\{0\}}$ such that

$$\mathfrak{M}_t\cap \operatorname{ran} A^{1/2} = \mathfrak{M}_t^{\perp}\cap\operatorname{ran} A^{1/2} = \{0\}\ \ \textit{for all}\, t\ne 0\ .$$

Moreover,

$$P_{\mathfrak{M}_{-t}^{\perp}} = U_{-t}P_{\mathfrak{M}_t}U_t,\ t\ne 0, \tag{3.19}$$

and therefore

$$\mathfrak{M}_{-t}^{\perp} = U_{-t}\mathfrak{M}_t,\ t\ne 0.$$

Proof. Consider $B_t = U_t A U_{-t}$. Note that by the Stone theorem U_t is of the form $U_t = \exp(itH)$, $t \in \mathbb{R}$, where generator H is a self-adjoint operator in $\mathcal{H}$. Since

$$\operatorname{ran} B_t^{1/2} \cap \operatorname{ran} A^{1/2} = \{0\},\ t \neq 0,$$

we get

$$\begin{aligned} A &= (A + B_t)^{1/2} P_t (A + B_t)^{1/2}, \\ B_t &= (A + B_t)^{1/2} (I - P_t)(A + B_t)^{1/2},\ t \neq 0, \end{aligned}$$

where $P_t := P_{\mathfrak{M}_t}$ are orthogonal projections on $\mathfrak{M}_t \subset \mathcal{H}$ for all $t \neq 0$. Then

$$A = U_{-t}(A + B_t)^{1/2}(I - P_t)(A + B_t)^{1/2} U_t. \tag{3.20}$$

By virtue of equality

$$U_{-t}(A + B_t) = (B_{-t} + A)U_{-t}$$

we get

$$U_{-t}(A + B_t)^k = (B_{-t} + A)^k U_{-t}$$

for all $k \in \mathbb{N}$. Hence

$$U_{-t}(A + B_t)^{1/2} = (B_{-t} + A)^{1/2} U_{-t},\ t \neq 0.$$

Then (3.20) yields

$$A = (B_{-t} + A)^{1/2} U_{-t}(I - P_t) U_t (B_{-t} + A)^{1/2},\ t \neq 0.$$

Since also

$$A = (A + B_{-t})^{1/2} P_{-t} (A + B_{-t})^{1/2},\ t \neq 0,$$

we obtain (3.19) with $\mathfrak{M}_t := \operatorname{ran} P_t$. □

Next we show that there exists *increasing* (*decreasing*) chains of subspaces possessing the trivial intersection property (3.12).

Theorem 3.9. *Let $A \in \mathbf{B}_0^+(\mathcal{H})$ and $\operatorname{ran} A \neq \mathcal{H}$. Then there is an increasing sequence $\mathfrak{N}_1 \subset \mathfrak{N}_2 \subset \ldots$ of subspaces in $\mathcal{H}$ such that*

(1) $\mathfrak{N}_k \cap \operatorname{ran} A^{1/2} = \mathfrak{N}_k^{\perp} \cap \operatorname{ran} A^{1/2} = \{0\}$ *for all* $k \in \mathbb{N}$,
(2) $\bigcap\limits_{k \in \mathbb{N}} \mathfrak{N}_k^{\perp} = \{0\}$,
(3) $s - \lim\limits_{k \to \infty} P_{\mathfrak{N}_k} = I_{\mathcal{H}}$.

Proof. (1) Choose $\mathfrak{M} \subset \mathcal{H}$ such that $\mathfrak{M} \cap \operatorname{ran} A^{1/2} = \mathfrak{M}^{\perp} \cap \operatorname{ran} A^{1/2} = \{0\}$ and define

$$A_1 := A^{1/2} P_{\mathfrak{M}} A^{1/2},\ A_2 := A_1^{1/2} P_{\mathfrak{M}} A_1^{1/2},\ \ldots A_k := A_{k-1}^{1/2} P_{\mathfrak{M}} A_k^{1/2},\ \ldots.$$

Then

$$A \geq A_1 \geq A_2 \geq \cdots, \tag{3.21}$$

and $\operatorname{ran} A_k^{1/2} = A_{k-1}^{1/2}\mathfrak{M} \subset \operatorname{ran} A_{k-1}^{1/2}$ for $k \in \mathbb{N}$. It follows that

$$\begin{aligned} &\operatorname{ran} A^{1/2} \supset \operatorname{ran} A_1^{1/2} \supset \operatorname{ran} A_2^{1/2} \supset \cdots, \\ &\mathfrak{M} \cap \operatorname{ran} A_k^{1/2} = \mathfrak{M}^{\perp} \cap \operatorname{ran} A_k^{1/2} = \{0\},\ k \in \mathbb{N}. \end{aligned}$$

In addition, by Proposition 3.2(1), $\operatorname{ran} A_1^{1/2} \cap \operatorname{ran} A = \{0\}$. Hence,

$$\operatorname{ran} A_k^{1/2} \cap \operatorname{ran} A = \{0\},\ k \in \mathbb{N}. \tag{3.22}$$

Now from Proposition 3.3 and (3.21) it follows by induction that

$$A_k = A^{1/2} P_k A^{1/2},\ k \in \mathbb{N}\ ,$$

where $\{P_k\}_{k\in\mathbb{N}}$ are orthogonal projections in $\mathcal{H}$ with $P_1 := P_{\mathfrak{M}}$ such that

$$P_1 \geq P_2 \geq \ldots \geq P_k \geq \ldots\ .$$

Since $\ker A_k = \{0\}$, one gets $\operatorname{ran}(I - P_k) \cap \operatorname{ran} A^{1/2} = \{0\}$. Equation (3.22) and Proposition 3.2 yield also that $\operatorname{ran} P_k \cap \operatorname{ran} A^{1/2} = \{0\}$. Set

$$\mathfrak{N}_1 = \mathfrak{M}^\perp,\ \mathfrak{N}_k = \operatorname{ran}(I - P_k),\ k \in \mathbb{N}.$$

Then we obtain

$$\mathfrak{M}^\perp = \mathfrak{N}_1 \subset \mathfrak{N}_2 \subset \ldots,\ \mathfrak{M} = \mathfrak{N}_1^\perp \supset \mathfrak{N}_2^\perp \supset \ldots\ .$$

(2) Now let us show that

$$\bigcap_{k\in\mathbb{N}} \mathfrak{N}_k^\perp = \{0\}.$$

Note first that the sequence $\{A_k\}_{k\geq 1} \subset \mathbf{B}^+(\mathcal{H})$ is non-increasing. So, it has the strong limit $A_0 := s-\lim_{k\to\infty} A_k$ and $\operatorname{ran} A_0^{1/2} \subset \operatorname{ran} A^{1/2}$. Therefore, from $A_k = A_{k-1}^{1/2} P_{\mathfrak{M}} A_k^{1/2}$, $k \in \mathbb{N}$ we obtain $A_0 = A_0^{1/2} P_{\mathfrak{M}} A_0^{1/2}$. Hence $\operatorname{ran} A_0^{1/2} \subset \mathfrak{M}$. On the other hand $\operatorname{ran} A_0^{1/2} \cap \mathfrak{M} = \{0\}$, which implies that operator $A_0 = 0$.

Suppose $f \in \bigcap\limits_{k\in\mathbb{N}} \mathfrak{N}_k^\perp$, i.e., $f = A_k^{1/2} f_k$. The equality $A_k = A_{k-1}^{1/2} P_{\mathfrak{M}} A_k^{1/2}$ and Proposition 3.2 imply that

$$||f_k|| = ||A^{-1/2} f|| \quad \text{for all} \quad k \in \mathbb{N}.$$

Since $s - \lim_{k\to\infty} A_k = 0$ and

$$(f, h) = (f_k, A_k^{1/2} h),\ k \in \mathbb{N},$$

we get $f = 0$. Thus $\bigcap\limits_{k\in\mathbb{N}} \mathfrak{N}_k^\perp = \{0\}$.

(3) Moreover, since $s - \lim_{k\to\infty} P_k = 0$, we also get $s - \lim_{k\to\infty} P_{\mathfrak{N}_k} = I_{\mathcal{H}}$. □

Note that Theorem 3.7 can be reformulated in terms of the operator ranges as follows.

Theorem 3.10. *Let operator range $\mathcal{R}$ be non-closed and dense in a Hilbert space $\mathcal{H}$. Then there is a subspace $\mathfrak{M} \subset \mathcal{H}$ such that*

$$\mathfrak{M} \cap \mathcal{R} = \mathfrak{M}^\perp \cap \mathcal{R} = \{0\}.$$

Proof. Let $A \in \mathbf{B}_0^+(\mathcal{H})$, and $\operatorname{ran} A^{1/2} = \mathcal{R}$. Then apply Theorem 3.7. □

Concequently, by applying Proposition 3.2 and Theorem 3.7 to $A = B^2$, where $B \in \mathbf{B}^+(\mathcal{H})$, one can now prove the Van Daele Theorem 1.2.

Corollary 3.11. *Let $\{F_j\}_{j=1}^n \in \mathbf{B}^+(\mathcal{H})$, $\ker \left(\sum\limits_{j=1}^n F_j\right) = \{0\}$, and $\operatorname{ran}\left(\sum\limits_{j=1}^n F_j\right) \neq \mathcal{H}$. Then there are infinitely many subspaces $\mathfrak{M}$ such that*

$$\mathfrak{M} \cap \operatorname{ran}\left(\sum_{j=1}^n F_j\right)^{1/2} = \mathfrak{M}^\perp \cap \operatorname{ran}\left(\sum_{j=1}^n F_j\right)^{1/2} = \{0\}.$$

In particular

$$\mathfrak{M}\cap\operatorname{ran}F_j^{1/2}=\mathfrak{M}^\perp\cap\operatorname{ran}F_j^{1/2}=\{0\}\ \textit{for all}\ j=1,2\ldots,n.$$

Corollary 3.12. *For arbitrary operator $A\in\mathbf{B}_0^+(\mathcal{H})$ with $\operatorname{ran}A\ne\mathcal{H}$ there exists infinitely many pairs $A_1,\ A_2\in\mathbf{B}^+(\mathcal{H})$ such that*

$$\begin{array}{l}A=A_1+A_2,\\ \ker A_1=\ker A_2=\{0\},\\ \operatorname{ran}A_1^{1/2}\cap\operatorname{ran}A_2^{1/2}=0,\\ \operatorname{ran}A_1^{1/2}\dot{+}\operatorname{ran}A_2^{1/2}=\operatorname{ran}A^{1/2}.\end{array}\tag{3.23}$$

Proof. Let $\mathfrak{M}$ be a subspace and $\mathfrak{M}\cap\operatorname{ran}A^{1/2}=\mathfrak{M}^\perp\cap\operatorname{ran}A^{1/2}=\{0\}$. Define two operators

$$A_1=A^{1/2}P_{\mathfrak{M}}A^{1/2},\ A_2=A^{1/2}P_{\mathfrak{M}^\perp}A^{1/2}.$$

Then equalities in (3.23) are satisfied. □

Since by definition for any operator $A\in\mathbf{B}^+(\mathcal{H})$ the set of all *extreme points* of the operator interval $[0,A]$ are of the form

$$\{A^{1/2}PA^{1/2}:\ P\text{ is an arbitrary orthogonal projection in }\mathcal{H}\},$$

see [34], the statement of Corollary 3.12 has the following interpretation:

There exists infinitely many pairs $\langle X,A-X\rangle$ of extreme points of the operator interval $[0,A]$ such that $\ker X=\ker(A-X)=\{0\}$. *Moreover, there are increasing* (*decreasing*) *sequences $\{X_n\}_{n\ge1}$ of such extreme points, which in addition have the property $s-\lim_{n\to\infty}X_n=A$ $(s-\lim_{n\to\infty}X_n=0)$.*

3.2. **Lifting of operators.** Let $A\in\mathbf{B}_0^+(\mathcal{H})$ with $\operatorname{ran}A\ne\mathcal{H}$. For a given subspace $\mathfrak{M}$, possessing the property (3.12), and for the corresponding orthogonal projection $P_{\mathfrak{M}}$, we are looking for existence of representation of the operator A in the form:

$$\begin{array}{l}A=T^{1/2}P_{\mathfrak{M}}T^{1/2},\\ T\in\mathbf{B}_0^+(\mathcal{H}),\\ \mathfrak{M}\cap\operatorname{ran}T^{1/2}=\mathfrak{M}^\perp\cap\operatorname{ran}T^{1/2}=\{0\}.\end{array}\tag{3.24}$$

We call this representation the *lifting of operator* A and we refer to the operator T as to the *lifting operator* for a given subspace $\mathfrak{M}$.

The following statement, which makes our concept of lifting nontrivial can be easily derived from Proposition 3.5 and Theorem 3.10.

Proposition 3.13. *Let $A\in\mathbf{B}_0^+(\mathcal{H})$ with $\operatorname{ran}A\ne\mathcal{H}$. Then operator A admits a lifting in the form*

$$A=T^{1/2}PT^{1/2},$$

where $T\in\mathbf{B}_0^+(\mathcal{H})$ and P is an orthogonal projection in $\mathcal{H}$ such that

$$\operatorname{ran}P\cap\operatorname{ran}T^{1/2}=\operatorname{ran}(I-P)\cap\operatorname{ran}T^{1/2}=\{0\}\ .$$

Notice that from the Proposition 3.13 one also obtains the triviality of intersections:

$$\operatorname{ran}P\cap\operatorname{ran}A^{1/2}=\{0\}\quad\text{and}\quad\operatorname{ran}(I-P)\cap\operatorname{ran}A^{1/2}=\{0\}.$$

For the following we need an auxiliary statement concerning the operator ranges, which we formulate as the lemma.

Lemma 3.14. *Let operator range $\mathcal{R}$ be dense in $\mathcal{H}$. Then there are operators $Y \in \mathbf{B}_0^+(\mathcal{H})$ such that*

$$\operatorname{ran} Y \cap \mathcal{R} = \{0\} \quad \text{and} \quad \operatorname{ran} Y^{1/2} \supset \mathcal{R} .$$

Proof. Let $Z \in \mathbf{B}^+(\mathcal{H})$ with $\mathcal{R} = \operatorname{ran} Z^{1/2}$. Since $\mathcal{R}$ is dense, we have $\ker Z = \{0\}$. Then by Proposition 2.6 one can find $X \in \mathbf{B}^+(\mathcal{H})$, such that

$$\ker X = \{0\} \quad \text{and} \quad \operatorname{ran} X^{1/2} \cap \operatorname{ran} Z^{1/2} = \{0\},$$

and $Z = (Z+X)^{1/2} P (Z+X)^{1/2}$, where P is an orthogonal projection. Set $Y := Z + X$. By construction we have $\operatorname{ran} Y^{1/2} \supset \operatorname{ran} Z^{1/2}$. Since $\ker X = \ker Z = \{0\}$, by Proposition 3.2 we get $\operatorname{ran} Y \cap \operatorname{ran} Z^{1/2} = \{0\}$ and the proof is completed. □

Theorem 3.15. *Let $A \in \mathbf{B}_0^+(\mathcal{H})$ and let $\mathfrak{M}$ be a subspace possessing* (3.12). *Suppose that the block operator-matrix A is of the form:*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{bmatrix} : \begin{matrix} \mathfrak{M} \\ \oplus \\ \mathfrak{M}^\perp \end{matrix} \to \begin{matrix} \mathfrak{M} \\ \oplus \\ \mathfrak{M}^\perp \end{matrix} .$$

Then operator A admits the lifting in the form (3.24) *for the subspace $\mathfrak{M}$ if and only if*

$$\operatorname{ran} A_{12} \subset \operatorname{ran} A_{11}^{3/4}. \tag{3.25}$$

Proof. By Corollary 3.6 the block operator-matrix A with respect to decomposition $\mathcal{H} = \mathfrak{M} \oplus \mathfrak{M}^\perp$ is of the form (3.8):

$$A = \begin{bmatrix} W^2 & WU \\ U^*W & U^*U \end{bmatrix} : \begin{matrix} \mathfrak{M} \\ \oplus \\ \mathfrak{M}^\perp \end{matrix} \to \begin{matrix} \mathfrak{M} \\ \oplus \\ \mathfrak{M}^\perp \end{matrix} ,$$

where the matrix entry are

$$A_{11} = W^2, \ A_{12} = WU, \ A_{12}^* = U^*W, \ A_{22} = U^*U.$$

Note that by (3.9), $\operatorname{ran} W \cap \operatorname{ran} U = \{0\}$, i.e., $\operatorname{ran} A_{11}^{1/2} \cap \operatorname{ran} (A_{11}^{-1/2} A_{12}) = \{0\}$.

Suppose that representation (3.24) is valid for some $T \in \mathbf{B}_0^+(\mathcal{H})$. By virtue of (2.9) the operator $T^{1/2}$ has a matrix form with respect to decomposition $\mathcal{H} = \mathfrak{M} \oplus \mathfrak{M}^\perp$:

$$T^{1/2} = \begin{bmatrix} X_{11} & X_{11}^{1/2} G X_{22}^{1/2} \\ X_{22}^{1/2} G^* X_{11}^{1/2} & X_{22} \end{bmatrix},$$

where $G \in \mathbf{B}(\mathfrak{M}^\perp, \mathfrak{M})$ is a contraction. Since $\ker T = \{0\}$, one has $\ker X_{11} = \{0\}$ and $\ker X_{22} = \{0\}$. Hence

$$\begin{bmatrix} W^2 & WU \\ U^*W & U^*U \end{bmatrix} = A = T^{1/2} P_{\mathfrak{M}} T^{1/2} = \begin{bmatrix} X_{11}^2 & X_{11} X_{12} \\ X_{12}^* X_{11} & X_{12}^* X_{12} \end{bmatrix},$$

where $X_{12} := X_{11}^{1/2} G X_{22}^{1/2}$ and consequently

$$X_{11} = W, \ X_{12} = U = W^{1/2} G X_{22}^{1/2}, \ \operatorname{ran} W^{1/2} \supset \operatorname{ran} U .$$

Therefore, the inclusion (3.25) holds.

Now assume (3.25) and define $M := W^{-1/2}U = A_{11}^{-3/4}A_{12}$. Since $\operatorname{ran} U \cap \operatorname{ran} W = \{0\}$, we get $\operatorname{ran} M \neq \mathcal{H}$. The latter and the equality $\ker U = \{0\}$ imply that $\operatorname{ran} M^*$ is dense in $\mathcal{H}$ but $\operatorname{ran} M^* \neq \mathcal{H}$. Now, by Lemma 3.14 one can find $Y \in \mathbf{B}_0^+(\mathfrak{M}^\perp)$ such that

$$\operatorname{ran} Y \cap \operatorname{ran} M^* = \{0\} \quad \text{and} \quad \operatorname{ran} Y^{1/2} \supset \operatorname{ran} M^*.$$

Define $Q := tY^{-1/2}M^*$, where $t > 0$ is such that $||Q|| \le 1$ and set

$$X_{22} := Y/t^2,\ G := Q^*.$$

Then $M^* = X_{22}^{1/2}G^*$, $M = GX_{22}^{1/2}$ and finally $U = W^{1/2}GX_{22}^{1/2}$.

Let us introduce

$$L := \begin{bmatrix} W & W^{1/2}GX_{22}^{1/2} \\ X_{22}^{1/2}G^*W^{1/2} & X_{22} \end{bmatrix} = \begin{bmatrix} W & U \\ U^* & X_{22} \end{bmatrix}.$$

Then $L \in \mathbf{B}_0^+(\mathcal{H})$ and the equality

$$A = LP_{\mathfrak{M}}L = \begin{bmatrix} W^2 & WU \\ U^*W & U^*U \end{bmatrix}$$

holds. If now we define $T := L^2$, then $A = T^{1/2}P_{\mathfrak{M}}T^{1/2}$, where

$$T^{1/2}\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} Wf_1 + Uf_2 \\ U^*f_1 + X_{22}f_2 \end{bmatrix}$$

$$= \begin{bmatrix} Wf_1 + Uf_2 \\ X_{22}^{1/2}G^*W^{1/2}f_1 + X_{22}f_2 \end{bmatrix} = \begin{bmatrix} Wf_1 + Uf_2 \\ M^*W^{1/2}f_1 + X_{22}f_2 \end{bmatrix}$$

Since $\operatorname{ran} U \cap \operatorname{ran} W = \{0\}$ and $\operatorname{ran} X_{22} \cap \operatorname{ran} M^* = \{0\}$, we obtain

$$\mathfrak{M} \cap \operatorname{ran} T^{1/2} = \mathfrak{M}^\perp \cap \operatorname{ran} T^{1/2} = \{0\},$$

which yields representation (3.24). □

The next statement follows from Corollary 3.6, Theorem 3.15, and (3.10), (3.11).

Corollary 3.16. *Let $A \in \mathbf{B}_0^+(\mathcal{H})$. Let $\mathfrak{M}$ be a subspace in $\mathcal{H}$ such that* $\dim \mathfrak{M} = \dim \mathfrak{M}^\perp = \infty$. *Then operator A admits the lifting in the form* (3.24) *for the subspace $\mathfrak{M}$ if and only if*

$$A = \begin{bmatrix} W^2 & WV\Phi \\ \Phi^*VW & \Phi^*V^2\Phi \end{bmatrix} : \begin{array}{c} \mathfrak{M} \\ \oplus \\ \mathfrak{M}^\perp \end{array} \to \begin{array}{c} \mathfrak{M} \\ \oplus \\ \mathfrak{M}^\perp \end{array},$$

where

$$(3.26)\qquad \begin{array}{l} W \in \mathbf{B}_0^+(\mathfrak{M}),\ V \in \mathbf{B}_0^+(\mathfrak{M}), \\ \Phi \quad \textit{unitarily maps} \quad \mathfrak{M}^\perp \quad \textit{onto} \quad \mathfrak{M}, \\ \operatorname{ran} V \cap \operatorname{ran} W = \{0\}, \\ \operatorname{ran} V \subset \operatorname{ran} W^{1/2}. \end{array}$$

Similarly, the operator A admits the lifting in the form

$$(3.27)\qquad \begin{array}{l} A = Q^{1/2}P_{\mathfrak{M}^\perp}Q^{1/2}, \\ Q \in \mathbf{B}_0^+(\mathcal{H}), \\ \mathfrak{M} \cap \operatorname{ran} Q^{1/2} = \mathfrak{M}^\perp \cap \operatorname{ran} Q^{1/2} = \{0\} \end{array}$$

if and only if

$$\begin{aligned} &\operatorname{ran} V \cap \operatorname{ran} W = \{0\},\\ &\operatorname{ran} V^{1/2} \supset \operatorname{ran} W. \end{aligned} \tag{3.28}$$

Finally, the operator A admits the lifting in the both forms (3.24) *and* (3.27) *if and only if*

$$\begin{aligned} &\operatorname{ran} V \cap \operatorname{ran} W = \{0\},\\ &\operatorname{ran} W^{1/2} \supset \operatorname{ran} V,\\ &\operatorname{ran} V^{1/2} \supset \operatorname{ran} W. \end{aligned} \tag{3.29}$$

One can resume the above observations as following:
Let $W \in \mathbf{B}_0^+(\mathfrak{M})$ with $\operatorname{ran} W \neq \mathfrak{M}$. Then there exists a subspace $\mathfrak{L} \subset \mathfrak{M}$ such that

$$\mathfrak{L} \cap \operatorname{ran} W^{1/2} = \mathfrak{L}^\perp \cap \operatorname{ran} W^{1/2} = \{0\}. \tag{3.30}$$

(a) Define the operator

$$V_1 := W^{1/2} P_{\mathfrak{L}} W^{1/2}.$$

Then one obtains that

$$\operatorname{ran} V_1^{1/2} \cap \operatorname{ran} W = \{0\} \quad \text{and} \quad \operatorname{ran} V_1 \subset \operatorname{ran} W^{1/2},$$

i.e., the operator V_1 satisfies (3.26), but it does *not* satisfy (3.28). This means that for any unitary mapping Φ of $\mathfrak{M}^\perp$ onto $\mathfrak{M}$ the operator

$$A_1 := \begin{bmatrix} W^2 & WV_1\Phi \\ \Phi^* V_1 W & \Phi^* V_1^2 \Phi \end{bmatrix} : \begin{array}{c} \mathfrak{M} \\ \oplus \\ \mathfrak{M}^\perp \end{array} \to \begin{array}{c} \mathfrak{M} \\ \oplus \\ \mathfrak{M}^\perp \end{array}$$

admits the lifting (3.24) by T, but it does not admit the lifting by Q in the form (3.27).
(b) Let us define

$$V_2 := W^{1/2}(I + P_{\mathfrak{L}})W^{1/2}.$$

Using (2.2) and the equality $\operatorname{ran}(I+P_{\mathfrak{L}}) = \mathfrak{M}$, we get that $\operatorname{ran} V_2^{1/2} = \operatorname{ran} W^{1/2}$. On the other hand if $V_2 x = Wy$, then $(I + P_{\mathfrak{L}})W^{1/2}x = W^{1/2}y$. It follows that $P_{\mathfrak{L}} W^{1/2} x = W^{1/2}(y - x)$. Condition (3.30) yields that $y = x = 0$. Hence, $\operatorname{ran} V_2 \cap \operatorname{ran} W = \{0\}$, i.e., the operator V_2 satisfies (3.29). Consequently, the operator

$$A_2 := \begin{bmatrix} W^2 & WV_2\Phi \\ \Phi^* V_2 W & \Phi^* V_2^2 \Phi \end{bmatrix} : \begin{array}{c} \mathfrak{M} \\ \oplus \\ \mathfrak{M}^\perp \end{array} \to \begin{array}{c} \mathfrak{M} \\ \oplus \\ \mathfrak{M}^\perp \end{array},$$

admits the lifting in the form (3.24) and in the form (3.27) for any unitary Φ .
(c) Choose $V \in \mathbf{B}_0^+(\mathfrak{M})$ such that $\operatorname{ran} V^{1/2} \cap \operatorname{ran} W^{1/2} = \{0\}$. Then operator V satisfies the condition $\operatorname{ran} V \cap \operatorname{ran} W = \{0\}$, see (3.11), but it does *not* satisfies both conditions (3.26) and (3.28). Therefore, operator A does *not* admits the lifting in the form (3.24) *and* in the form (3.27). This example indicates a *limit* for application of our method.

3.3. **Applications to unbounded operators.** First we present here an extended version and a new proof of the Schmüdgen Theorem 1.4. The both follow from our results in Section 3.1 and 3.2.

Theorem 3.17. *Let H be a closed unbounded densely defined linear operator in a Hilbert space $\mathcal{H}$. Then*

(1) *there exists a subspace $\mathfrak{M}$ of $\mathcal{H}$ such that*

$$\mathfrak{M} \cap \operatorname{dom} H = \mathfrak{M}^{\perp} \cap \operatorname{dom} H = \{0\} \ , \tag{3.31}$$

moreover, there exists uncountably many of them ,

(2) *there exists a fundamental symmetry J in $\mathcal{H}$ such that*

$$J \operatorname{dom} H \cap \operatorname{dom} H = \{0\}, \tag{3.32}$$

moreover, there exists uncountably many of them.

Proof. Let $A = (H^*H + I)^{-1}$. Then $A \in \mathbf{B}^+(\mathcal{H})$ and $\mathcal{R} := \operatorname{ran} A^{1/2} = \operatorname{dom} H$. By Theorem 3.7, Proposition 3.8, and Theorem 3.10 there exists uncountable set of subspaces $\mathfrak{M}$ of $\mathcal{H}$ satisfying (3.31). Therefore, combining this observation with Proposition 3.1 we deduce that there exists uncountable set of fundamental symmetries J satisfying (3.32). □

Note that by virtue of Theorem 3.10 and Proposition 3.8 there exists a one-parameter family $\{\mathfrak{M}_t\}_{t\in\mathbb{R}}$ of subspaces and operators $\{J_t := (2P_{\mathfrak{M}_t} - I)\}_{t\in\mathbb{R}}$ satisfying respectively (3.31) and (3.32). Then besides Proposition 3.1 we can formulate the following version of the von Neumann Theorem 2.2.

Corollary 3.18. *For any unbounded self-adjoint operator H in a Hilbert space $\mathcal{H}$ there exists a one-parameter family $\{U_t\}_{t\in\mathbb{R}}$ of unitary operators with the property*

$$\operatorname{dom} H \cap \operatorname{dom} (U_t^* H U_t) = \{0\} \ .$$

Moreover, one can find a strongly continuous family $\{J_t\}_{t\in\mathbb{R}}$ of fundamental symmetries (i.e. self-adjoint and unitary operators) such that

$$\operatorname{dom} H \cap \operatorname{dom} (J_t H J_t) = \{0\} \ .$$

Corollary 3.19. *Let $T_1, \ldots, T_n$ be closed unbounded densely defined linear operators in a Hilbert space $\mathcal{H}$. Then there exists infinitely many subspaces $\mathfrak{M}$ in $\mathcal{H}$ such that*

$$\mathfrak{M} \cap \left(\sum_{j=1}^{n} \operatorname{dom} T_k\right) = \mathfrak{M}^{\perp} \cap \left(\sum_{j=1}^{n} \operatorname{dom} T_k\right) = \{0\}.$$

In particular

$$\mathfrak{M} \cap \operatorname{dom} T_j = \mathfrak{M}^{\perp} \cap \operatorname{dom} T_j = \{0\} \textit{ for all } j = 1, \ldots, n.$$

Let T be non-negative self-adjoint operator in $\mathcal{H}$. As it is well-known [20], [24] the sesquilinear form (Tu, v), $u, v \in \operatorname{dom} T$ admits a closure and we (following Kreĭn [24]) denote this closure by $T[\cdot, \cdot]$ and its domain by $\mathcal{D}[T]$ (see Subsection 2.6). Then by the *second representation theorem* [20]

$$\mathcal{D}[T] = \operatorname{dom} T^{1/2} \ \text{ and } \ T[f, g] = (T^{1/2} f, T^{1/2} g), \ f, g \in \operatorname{dom} T^{1/2}.$$

The linear manifold $\mathcal{D}[T](=\operatorname{dom} T^{1/2})$ is the Hilbert space with respect to the graph inner product

$$(f,g)_{T^{1/2}} = T[f,g] + (f,g). \tag{3.33}$$

The fractional-linear transformation

$$S = (I-T)(I+T)^{-1},\ T = (I-S)(I+S)^{-1}$$

gives a one-to-one correspondence between the set of all non-negative self-adjoint operators T and the set of all self-adjoint contractions S such that $\ker(S+I)=\{0\}$, see [24]. Then one can easily derive [6] that

$$\begin{array}{l}\mathcal{D}[T] = \operatorname{ran}(I+S)^{1/2},\\ T[u,v] = -(u,v) + 2\left((I+S)^{-1/2}u, (I+S)^{-1/2}v\right), \quad u,v\in\mathcal{D}[T].\end{array} \tag{3.34}$$

The next application of our approach is the following theorem and the corresponding remarks.

Theorem 3.20. *Let T be unbounded non-negative self-adjoint operator in Hilbert space $\mathcal{H}$. Then there are infinitely many pairs $\langle T_1, T_2\rangle$ of unbounded non-negative self-adjoint operators such that*

1. $\operatorname{dom} T_1^{1/2}\cap\operatorname{dom} T = \operatorname{dom} T_2^{1/2}\cap\operatorname{dom} T = \{0\}$;
2. $\operatorname{dom} T_k^{1/2}\subset\operatorname{dom} T^{1/2}$, *and* $\|T_k^{1/2}g\| = \|T^{1/2}g\|$ *for all* $g\in\operatorname{dom} T_k^{1/2}$, $k=1,2$;
3. $\operatorname{dom} T_1^{1/2}\cap\operatorname{dom} T_2^{1/2} = \{0\}$;
4. *the Hilbert space $\mathcal{D}[T]$ admits the orthogonal decomposition $\mathcal{D}[T]=\mathcal{D}[T_1]\oplus_{T^{1/2}}\mathcal{D}[T_2]$ with respect to the inner product (3.33).*

Proof. Let $S=(I-T)(I+T)^{-1}$ and let $\mathfrak{M}$ be a subspace in $\mathcal{H}$ such that (see Corollary 3.17)

$$\mathfrak{M}\cap\operatorname{dom} T^{1/2} = \mathfrak{M}^\perp\cap\operatorname{dom} T^{1/2} = \{0\}.$$

We define

$$S_1 = (I+S)^{1/2}P_{\mathfrak{M}}(I+S)^{1/2} - I\ ,\quad S_2 = (I+S)^{1/2}P_{\mathfrak{M}^\perp}(I+S)^{1/2} - I. \tag{3.35}$$

The operators S_1 and S_2 are self-adjoint contractions with $\ker(S_k+I)=\{0\}$, $k=1,2$. Let

$$T_k = (I-S_k)(I+S_k)^{-1},\ k=1,2.$$

Then T_1 and T_2 are non-negative self-adjoint operators. Using (3.34) and (3.35) we have

$$\operatorname{dom} T^{1/2} = \operatorname{ran}(I+S)^{1/2},$$
$$\operatorname{dom} T_1^{1/2} = (I+S)^{1/2}\mathfrak{M},\ \operatorname{dom} T_2^{1/2} = (I+S)^{1/2}\mathfrak{M}^\perp.$$

Notice that by definitions

$$\begin{array}{l}\operatorname{dom} T_1^{1/2}\cap\operatorname{dom} T_2^{1/2} = \{0\},\\ \operatorname{dom} T_1^{1/2}\dot{+}\operatorname{dom} T_2^{1/2} = \operatorname{dom} T^{1/2}.\end{array}$$

Suppose $(I+S_1)^{1/2}u = (I+S)f$, i.e., $(I+S)^{1/2}x = (I+S)f$ for some $x\in\mathfrak{M}$. Hence $x=(I+S)^{1/2}f$. But $\operatorname{ran}\mathfrak{M}\cap(I+S)^{1/2}=\{0\}$. This means that $u=f=0$ and, therefore, $\operatorname{dom} T_1^{1/2}\cap\operatorname{dom} T=\{0\}$. Similarly $\operatorname{dom} T_2^{1/2}\cap\operatorname{dom} T=\{0\}$.

From $I+S_1=(I+S)^{1/2}P_{\mathfrak{M}}(I+S)^{1/2}$ we obtain

$$(I+S_1)^{1/2}h=(I+S)^{1/2}\mathcal{U}h,\ h\in H,$$

where $\mathcal{U}$ is unitary operator from $\mathcal{H}$ onto $\mathfrak{M}(=\operatorname{ran}P_{\mathfrak{M}})$. Hence

$$(I+S)^{-1/2}g=\mathcal{U}(I+S_1)^{-1/2}g \quad \text{for all} \quad g\in\operatorname{ran}(I+S_1)^{1/2}=\operatorname{dom}T_1^{1/2}.$$

Thus,

$$||(I+S_1)^{-1/2}g||^2=||(I+S)^{-1/2}g||^2,\ g\in\operatorname{ran}(I+S_1)^{1/2}. \tag{3.36}$$

Now (3.34) and (3.36) yield $||T_1^{1/2}g||=||T^{1/2}g||$ for all $g\in\operatorname{dom}T_1^{1/2}$. Similarly $||T_2^{1/2}g||=||T^{1/2}g||$ for all $g\in\operatorname{dom}T_2^{1/2}$.

Let $u\in\operatorname{dom}T_1^{1/2}$, $v\in\operatorname{dom}T_2^{1/2}$. Then

$$\begin{aligned}u&=(I+S)^{1/2}f,\ f\in\mathfrak{M},\\ v&=(I+S)^{1/2}h,\ h\in\mathfrak{M}^\perp.\end{aligned}$$

From (3.33) and (3.34) we get $(u,v)_{T^{1/2}}=0$. This yields the orthogonal decomposition $\mathcal{D}[T]=\mathcal{D}[T_1]\oplus_{T^{1/2}}\mathcal{D}[T_2]$ and the proof is completed. □

Remark 3.21. *From the proof one can also find the expressions of T_1 and T_2 via T:*

$$\begin{aligned}T_1&=\left((I+T)^{-1/2}P_{\mathfrak{M}}(I+T)^{-1/2}\right)^{-1}-I,\\ T_2&=\left((I+T)^{-1/2}P_{\mathfrak{M}^\perp}(I+T)^{-1/2}\right)^{-1}-I.\end{aligned}$$

Then

$$(I+T_1)^{-1}+(I+T_2)^{-1}=(I+T)^{-1}.$$

This means that any vector $f_T\in\operatorname{dom}T$ admits a unique decomposition

$$f_T=f_{T_1}+f_{T_2},$$

where $f_{T_1}\in\operatorname{dom}T_1$ and $f_{T_2}\in\operatorname{dom}T_2$, although

$$\operatorname{dom}T\cap\operatorname{dom}T_1=\operatorname{dom}T\cap\operatorname{dom}T_2=\{0\}.$$

Here $f_{T_1}=(I+T_1)^{-1}(I+T)f_T$, $f_{T_2}=(I+T_2)^{-1}(I+T)f_T$. In addition, the following equalities are valid:

$$\begin{aligned}\operatorname{dom}T_1^{1/2}&=(I+T)^{-1/2}\mathfrak{M},\\ \operatorname{dom}T_2^{1/2}&=(I+T)^{-1/2}\mathfrak{M}^\perp.\end{aligned}$$

Remark 3.22. (a) *Theorem 3.20 yields also that* $\ker T_1=\ker T_2=\{0\}$. *Suppose that $f\neq 0$ and $T_1f=0$, then (2) implies $Tf=0$, i.e., $\ker T_1\subseteq\ker T$. This gives* $\operatorname{dom}T_1^{1/2}\cap\operatorname{dom}T\neq\{0\}$, *which contradicts to (1).*
(b) *The equalities* (2.1) *and* (3.35) *imply that*

$$\operatorname{ran}T_1^{1/2}=\operatorname{ran}T^{1/2}+\operatorname{dom}T_2^{1/2},\ \operatorname{ran}T_2^{1/2}=\operatorname{ran}T^{1/2}+\operatorname{dom}T_1^{1/2}.$$

In particular

$$\operatorname{ran}T_1^{1/2}\supseteq\operatorname{ran}T^{1/2},\ \operatorname{ran}T_2^{1/2}\supseteq\operatorname{ran}T^{1/2}.$$

Remark 3.23. (a) *By the properties (2) and (3) the form-sum $T_k\dot{+}T$ is equal to operator $2T_k$, for $k=1,2$. Then the Lie-Trotter-Kato product formula* [21], [22] *implies the strong convergence*

$$s-\lim_{n\to\infty}\left(\exp(-tT/n)\exp(-tT_k/n)\right)^n=\exp(-2tT_k),\ t\ge 0,\ k=1,2.$$

(b) *Since* $\operatorname{dom}T_1^{1/2}\cap\operatorname{dom}T_2^{1/2}=\{0\}$, *the Kato theorem* [21, Theorem 1] *yields*

$$s-\lim_{n\to\infty}\left(\exp(-tT_1/n)\exp(-tT_2/n)\right)^n=s-\lim_{n\to\infty}\left(\exp(-tT_2/n)\exp(-tT_1/n)\right)^n=0.$$

Note also that Theorem 3.20 and Remark 3.21 yield the following statement.

Corollary 3.24. *Let T be unbounded non-negative self-adjoint operator in Hilbert space $\mathcal{H}$. Then for each natural number n there exists n unbounded non-negative self-adjoint operators $\{T_k\}_{k=1}^n$ such that*

1. $\operatorname{dom}T_k^{1/2}\cap\operatorname{dom}T=\{0\}$, $k=1,2,\ldots,n$,
2. *if $k\ne j$, then* $\operatorname{dom}T_k^{1/2}\cap\operatorname{dom}T_j^{1/2}=\{0\}$,
3. *the form $T_k[\cdot,\cdot]$ is a closed restrictions of the form $T[\cdot,\cdot]$*,
4. $\mathcal{D}[T]=\mathcal{D}[T_1]\oplus_{T^{1/2}}\mathcal{D}[T_2]\oplus_{T^{1/2}}\cdots\oplus_{T^{1/2}}\mathcal{D}[T_n]$,
5. $(T+I)^{-1}=(T_1+I)^{-1}+(T_2+I)^{-1}+\cdots+(T_n+I)^{-1}$.

Theorem 3.25. *Let T_1 be unbounded non-negative self-adjoint operator in $\mathcal{H}$ with $\ker T_1=\{0\}$. Then one can always find two non-negative self-adjoint operators T_2 and T such that conditions (1)–(4) of Theorem 3.20 are satisfied.*

Proof. Let

$$S_1:=(I-T_1)(I+T_1)^{-1},\ A_1=\frac{1}{2}\,(I+S_1).$$

Then $0\le A_1\le I$, $\ker A_1=\ker(I-A_1)=\{0\}$. By Theorem 2.3 there exists $X\in\mathbf{B}_0^+(\mathcal{H})$ such that $\operatorname{ran}X^{1/2}\cap\operatorname{ran}A_1^{1/2}=\{0\}$, $0\le X\le I$. Set

$$B:=(I-A_1)^{1/2}X(I-A_1)^{1/2}.$$

Then $\ker B=\{0\}$, $0\le A_1+B\le I$, $\ker(A_1+B)=\{0\}$, and the equalities

$$\operatorname{ran}B^{1/2}=(I-A_1)^{1/2}\operatorname{ran}X^{1/2},\ \operatorname{ran}(A_1-A_1^2)^{1/2}=\operatorname{ran}A_1^{1/2}\cap\operatorname{ran}(I-A_1)^{1/2},$$
$$\operatorname{ran}X^{1/2}\cap\operatorname{ran}A_1^{1/2}=\{0\}$$

imply $\operatorname{ran}B^{1/2}\cap\operatorname{ran}A_1^{1/2}=\{0\}$. Hence

$$A_1=(A_1+B)^{1/2}P(A_1+B)^{1/2},\ B=(A_1+B)^{1/2}(I-P)(A_1+B)^{1/2}\ ,$$

where P is some orthogonal projection in $\mathcal{H}$, see Proposition 2.6. Then define

$$\begin{array}{l}S:=2(A_1+B)-I,\ S_2:=2B-I,\\ T:=(I-S)(I+S)^{-1},\ T_2:=(I-S_2)(I+S_2)^{-1}.\end{array}$$

Since $I+S_1=(I+S)^{1/2}P(I+S)^{1/2}$ and $I+S_2=(I+S)^{1/2}(I-P)(I+S)^{1/2}$, one follows arguments used in Theorem 3.20 to complete the proof. ☐

The next statement is extension of Theorem 3.20 to the *infinite* family of operator pairs.

Theorem 3.26. *Let T be unbounded non-negative self-adjoint operator in Hilbert space $\mathcal{H}$. Then there are pairs of families $\langle\{T_{1,j}\}_{j\in\mathbb{N}}, \{T_{2,k}\}_{k\in\mathbb{N}}\rangle$ of unbounded non-negative self-adjoint operators possessing the following properties:*

(1) $\operatorname{dom} T^{1/2} \supset \operatorname{dom} T_{1,1}^{1/2} \supset \operatorname{dom} T_{1,2}^{1/2} \supset \cdots \supset \operatorname{dom} T_{1,j}^{1/2} \supset \cdots$,
(2) $\bigcap\limits_{j\in\mathbb{N}} \operatorname{dom} T_{1,j}^{1/2} = \{0\}$,
(3) $\operatorname{dom} T_{1,j}^{1/2} \cap \operatorname{dom} T = \{0\}$ $(T_{1,0} = T_{2,0} = T)$,
(4) $\operatorname{dom} T_{2,1}^{1/2} \subset \operatorname{dom} T_{2,2}^{1/2} \subset \cdots \subset \operatorname{dom} T_{2,j}^{1/2} \subset \cdots \subset \operatorname{dom} T^{1/2}$,
(5) $\operatorname{dom} T_{2,j}^{1/2} \cap \operatorname{dom} T = \{0\}$ *for all* $j \in \mathbb{N}$,
(6) *the sesquilinear forms* $T_{1,j}[\cdot,\cdot]$ *and* $T_{2,j}[\cdot,\cdot]$ *are closed restrictions of the form* $T[\cdot,\cdot]$ *for each* $j \in \mathbb{N}$,
(7) $\mathcal{D}[T] = \mathcal{D}[T_{1,j}] \oplus_{T^{1/2}} \mathcal{D}[T_{2,j}]$ *for each* $j \in \mathbb{N}$,
(8) $s-\lim\limits_{j\to\infty} (T_{1,j} - \lambda I)^{-1} = 0$, *for* $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$,
(9) $s-\lim\limits_{j\to\infty} (T_{2,j} - \lambda I)^{-1} = (T - \lambda I)^{-1}$, *for* $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$,
(10) $s-\lim\limits_{j\to\infty} \exp(-zT_{1,j}) = 0$ *for all* z, $\operatorname{Re} z > 0$,
(11) $s-\lim\limits_{j\to\infty} \exp(-zT_{2,j}) = \exp(-zT)$ *for all* z, $\operatorname{Re} z \geq 0$.

Proof. Let $S := (I-T)(I+T)^{-1}$. Then by Theorem 3.9 there is an increasing sequence $\mathfrak{N}_1 \subset \mathfrak{N}_2 \subset \ldots$ of subspaces in $\mathcal{H}$ such that

(1) $\mathfrak{N}_k \cap \operatorname{ran}(I+S)^{1/2} = \mathfrak{N}_k^\perp \cap \operatorname{ran}(I+S)^{1/2} = \{0\}$ for all $k \in \mathbb{N}$,
(2) $\bigcap\limits_{k\in\mathbb{N}} \mathfrak{N}_k^\perp = \{0\}$,
(3) $s-\lim\limits_{k\to\infty} P_{\mathfrak{N}_k} = I_{\mathcal{H}}$.

Then we define

$$S_{1,j} := (I+S)^{1/2} P_{\mathfrak{N}_j^\perp} (I+S)^{1/2} - I \quad\text{and}\quad S_{2,j} := (I+S)^{1/2} P_{\mathfrak{N}_j} (I+S)^{1/2} - I,\ j \in \mathbb{N}.$$

Due to $\mathfrak{N}_1^\perp \supset \mathfrak{N}_2^\perp \supset \ldots$ and $\mathfrak{N}_1 \subset \mathfrak{N}_2 \subset \ldots$ one obtains

$$I+S \geq I+S_{1,1} \geq I+S_{1,2} \geq \ldots,\ I+S_{2,1} \leq I+S_{2,2} \leq \ldots \leq I+S.$$

Hence for $j \in \mathbb{N}$:

$$\begin{aligned}&\operatorname{ran}(I+S_{1,j+1})^{1/2} \subset \operatorname{ran}(I+S_{1,j})^{1/2} \subset \operatorname{ran}(I+S)^{1/2},\\ &\operatorname{ran}(I+S_{2,j})^{1/2} \subset \operatorname{ran}(I+S_{2,j+1})^{1/2} \subset \operatorname{ran}(I+S)^{1/2}.\end{aligned}$$

Consequently, we get for $j, l, k \in \mathbb{N}$:

$$\begin{aligned}&\operatorname{ran}(I+S_{1,j})^{1/2} \cap \mathfrak{N}_k = \operatorname{ran}(I+S_{1,j})^{1/2} \cap \mathfrak{N}_k^\perp = \{0\},\\ &\operatorname{ran}(I+S_{2,l})^{1/2} \cap \mathfrak{N}_k = \operatorname{ran}(I+S_{2,l})^{1/2} \cap \mathfrak{N}_k^\perp = \{0\}.\end{aligned}$$

Since $s-\lim\limits_{j\to\infty} P_{\mathfrak{N}_j} = I$, one also has

$$s-\lim_{j\to\infty}(I+S_{1,j}) = 0 \quad\text{and}\quad s-\lim_{j\to\infty}(I+S_{2,j}) = I+S\ .$$

Now we define for $j \in \mathbb{N}$:

$$T_{1,j} := (I-S_{1,j})(I+S_{1,j})^{-1} \quad\text{and}\quad T_{2,j} := (I-S_{2,j})(I+S_{2,j})^{-1}.$$

Then $\{T_{1,j}\}$ and $\{T_{2,j}\}$ are non-negative self-adjoint operators, such that (see Theorem 3.20 and Theorem 3.9)

$$\operatorname{dom} T_{1,j}^{1/2} \cap \operatorname{dom} T = \{0\},\ \operatorname{dom} T_{2,j}^{1/2} \cap \operatorname{dom} T = \{0\},\ j \in \mathbb{N},$$

and properties (1)–(7) hold true. Since $(T_{1,j}+I)^{-1} = 2(I+S_{1,j})$ and $(T_{2,j}+I)^{-1} = 2(I+S_{2,j})$, we obtain

$$s - \lim_{j\to\infty} (T_{1,j} + I)^{-1} = 0 \quad \text{and} \quad s - \lim_{j\to\infty} (T_{2,j} + I)^{-1} = 2(I + S) = (I + T)^{-1}.$$

This implies (see [20, Chapter VIII, Theorem 1.3]) that

$$\begin{aligned} &s - \lim_{j\to\infty} (T_{1,j} - \lambda I)^{-1} = 0,\\ &s - \lim_{j\to\infty} (T_{2,j} - \lambda I)^{-1} = (T - \lambda I)^{-1} \end{aligned}$$

for $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$.

In order to prove for all z, $\operatorname{Re} z > 0$, the limit $s - \lim_{j\to\infty} \exp(-zT_{1,j}) = 0$ we use the Euler approximation of the one-parameter semigroup $\{\exp(-tA)\}_{t\geq 0}$ with $m-\alpha$ sectorial ($\alpha \in [0, \pi/2)$) generator A in the operator-norm topology [9],[41]:

$$|| \exp(-tA) - (I + tA/n)^{-n}|| \leq \frac{K_\alpha}{n \cos^2 \alpha},\ t \geq 0,\ n \in \mathbb{N},$$

Here K_α depends only on α, see [14].

Let $\widetilde{T}$ be non-negative self-adjoint operator and $\operatorname{Re} z \geq 0$. Then for $z = t\, e^{i\varphi}$ and $|\varphi| \in [0, \pi/2)$ the operator $A = e^{i\varphi}\widetilde{T}$ is $m - |\varphi|$-sectorial generator. Put $\widetilde{T} = T_{1,j}$. Then

$$\begin{aligned} || \exp(-zT_{1,j})f|| &\leq || \left(\exp(-t(e^{i\varphi}T_{1,j})) - (I + t(e^{i\varphi}T_{1,j})/n)^{-n}\right) f||\\ &\qquad + ||(I + t(e^{i\varphi}T_{1,j})/n)^{-n} f||\\ &\leq \frac{K_{|\varphi|}}{n \cos^2 \varphi} ||f|| + ||(I + t(e^{i\varphi}T_{1,j})/n)^{-n} f||\\ &\qquad\leq \frac{K_{|\varphi|}}{n \cos^2 \varphi} ||f|| + C(n, \varphi, t)||(I + \frac{te^{i\varphi}}{n} T_{1,j})^{-1} f||, \end{aligned}$$

for any $f \in \mathcal{H}$ and some constant $C(n, \varphi, t)$. Since above it was established that for each n, φ, f, and $t > 0$

$$\lim_{j\to\infty} ||(I + \frac{te^{i\varphi}}{n} T_{1,j})^{-1} f|| = 0\ ,$$

we obtain $s - \lim_{j\to\infty} \exp(-zT_{1,j}) = 0$.

Applying now the Trotter– Kato approximation theorem [18, Chapter III, Section 4.9], we obtain

$$s - \lim_{j\to\infty} \exp(-zT_{2,j}) = \exp(-zT) \text{ for all } z,\ \operatorname{Re} z \geq 0\ ,$$

and the end of the proof. ☐

Remark 3.27. 1) *The inclusions* $\operatorname{dom} T_{1,j-1}^{1/2} \supset \operatorname{dom} T_{1,j}^{1/2}$ *and equalities* $||T_{1,j}^{1/2} f|| = ||T^{1/2} f||$ *for all* $f \in \operatorname{dom} T_{1,j}^{1/2}$ *and all* $j \in \mathbb{N}$ *mean that*

$$T \leq T_{1,1} \leq T_{1,2} \leq \ldots \leq T_{1,k} \leq \ldots\ ,$$

in the sense of associated closed quadratic forms [20]*. It is proved by T. Kato* [21, Lemma 1]*, that if H_j for $j \in \mathbb{N}$ are self-adjoint operators in $\mathcal{H}$ such that $I_{\mathcal{H}} \le H_1 \le H_2 \le \dots$ and*

$$\mathcal{D}_0 = \{u \in \mathcal{H} : u \in \bigcap_{j\in\mathbb{N}} \operatorname{dom} H_j^{1/2},\ \sup_j ||H_j^{1/2}u|| < \infty\},$$

then $\lim\limits_{j\to\infty} H_j^{-1}v = 0$ for all $v \perp \mathcal{D}_0$.

2) *Let $\mathbf{H} := \{\langle 0, h\rangle\,,\ h \in \mathcal{H}\}$. Then $\mathbf{H}$ is a self-adjoint linear relation* [4]*. The resolvent $(\mathbf{H}-\lambda I)^{-1}$: $\mathcal{H} \to \mathcal{H}$ is identically zero operator. The equality $s-\lim\limits_{j\to\infty}(T_{1,j}-\lambda I)^{-1} = 0$ means that in the strong resolvent limit sense* [20] *the sequence of operators $\{T_{1,j}\}_{j\in\mathbb{N}}$ converges to* $\mathbf{H}$.

3) *From Remark 3.22 it follows that for all $j \in \mathbb{N}$:*

$$\operatorname{ran} T_{1,j}^{1/2} = \operatorname{ran} T^{1/2} + \operatorname{dom} T_{2,j}^{1/2} \quad \text{and} \quad \operatorname{ran} T_{2,j}^{1/2} = \operatorname{ran} T^{1/2} + \operatorname{dom} T_{1,j}^{1/2}.$$

In particular,

$$\begin{array}{l} \operatorname{ran} T^{1/2} \subseteq \operatorname{ran} T_{1,1}^{1/2} \subseteq \operatorname{ran} T_{1,2}^{1/2} \subseteq \dots,\ T^{-1} \ge T_{1,1}^{-1} \ge T_{1,2}^{-1} \ge \dots, \\ \operatorname{ran} T_{2,1}^{1/2} \supseteq \operatorname{ran} T_{2,2}^{1/2} \supseteq \dots \supseteq \operatorname{ran} T^{1/2},\ T_{2,1}^{-1} \le T_{2,2}^{-1} \le \dots \le T^{-1}, \end{array}$$

and

(1) $s-\lim\limits_{j\to\infty}(T_{1,j}^{-1} - \lambda I)^{-1} = -\lambda^{-1}I$,
(2) $s-\lim\limits_{j\to\infty} e^{-zT_{1,j}^{-1}} = I$, *for all z,* $\operatorname{Re} z > 0$,
(3) $s-\lim\limits_{j\to\infty}(T_{2,j}^{-1} - \lambda I)^{-1} = (T^{-1} - \lambda I)^{-1}$, *for $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$,* (*note that T^{-1} is, in general, a linear relation*) ,
(4) $\lim\limits_{j\to\infty} T_{2,j}^{-1}[u, v] = T^{-1}[u, v]$ *for all $u, v \in \operatorname{ran} T^{1/2}$.*

3.4. **Beyond the Van Daele–Schmüdgen and the Brasche–Neidhardt theorems.** We start by theorem, which is a weaker (but useful) version of the Van Daele Theorem 1.2.

Theorem 3.28. *Let B be a non-negative unbounded self-adjoint operator in a Hilbert space $\mathcal{H}$. Suppose that* $\ker B = \{0\}$ *and* $\operatorname{ran} B \ne \mathcal{H}$*. Then there exists two linear manifolds $\mathfrak{D}_1$ and $\mathfrak{D}_2$ possessing the following properties:*

(1) $\mathfrak{D}_1 \dot{+} \mathfrak{D}_2 = \operatorname{dom} B$,
(2) *$B\,\mathfrak{D}_1$ and $B\,\mathfrak{D}_2$ are dense in $\mathcal{H}$.*

Moreover, one can choose $\mathfrak{D}_1$ and $\mathfrak{D}_2$ such that $\mathfrak{D}_1 \oplus_B \mathfrak{D}_2 = \operatorname{dom} B$.

Proof. Set $S := (I - B^2)(I + B^2)^{-1}$. Then

$$B^2 = (I - S)(I + S)^{-1},\ B = (I - S)^{1/2}(I + S)^{-1/2}.$$

Hence

$$\operatorname{dom} B = \operatorname{ran}(I + S)^{1/2},\ B(I + S)^{1/2}h = (I - S)^{1/2}h,\ h \in \mathcal{H}.$$

Since $\ker(I - S) = \{0\}$ and $\operatorname{ran}(I - S) \ne \mathcal{H}$, there exists a subspace $\mathfrak{M}$ of $\mathcal{H}$ such that

$$\mathfrak{M} \cap \operatorname{ran}(I - S)^{1/2} = \mathfrak{M}^\perp \cap \operatorname{ran}(I - S)^{1/2} = \{0\}. \tag{3.37}$$

Let

$$\mathfrak{D}_1 := (I + S)^{1/2}\mathfrak{M} \quad \text{and} \quad \mathfrak{D}_2 := (I + S)^{1/2}\mathfrak{M}^\perp.$$

It follows then from (3.33) and (3.34) that

$$\mathfrak{D}_1 \oplus_B \mathfrak{D}_2 = \operatorname{dom} B.$$

Since $B\,\mathfrak{D}_1 = (I-S)^{1/2}\mathfrak{M}$, $B\,\mathfrak{D}_2 = (I-S)^{1/2}\mathfrak{M}^\perp$, and (3.37) holds, we conclude that the sets $B\,\mathfrak{D}_1$ and $B\,\mathfrak{D}_2$ are dense in $\mathcal{H}$. □

In the following theorem our approach elucidates the property of products and squares of unbounded operators.

Theorem 3.29. *Let T and $\langle T_1, T_2\rangle$ be as in Theorem 3.20. Define non-negative self-adjoint operator $\mathcal{L} := T^{1/2}$ together with two of its densely defined symmetric restrictions*

$$\dot{\mathcal{L}}_1 := T^{1/2}{\upharpoonright}\operatorname{dom} T_1^{1/2} \quad \text{and} \quad \dot{\mathcal{L}}_2 := T^{1/2}{\upharpoonright}\operatorname{dom} T_2^{1/2}.$$

Then

(1) *operators $\dot{\mathcal{L}}_1$ and $\dot{\mathcal{L}}_2$ are closed,*
(2) $\operatorname{dom}\dot{\mathcal{L}}_1 \cap \operatorname{dom}\mathcal{L}^2 = \operatorname{dom}\dot{\mathcal{L}}_2 \cap \operatorname{dom}\mathcal{L}^2 = \{0\}$,
(3) $\operatorname{dom}\dot{\mathcal{L}}_1 \cap \operatorname{dom}\dot{\mathcal{L}}_2 = \{0\}$ *and* $\operatorname{dom}\mathcal{L} = \operatorname{dom}\dot{\mathcal{L}}_1 \dot{+} \operatorname{dom}\dot{\mathcal{L}}_2$,
(4) $\operatorname{dom}(\mathcal{L}\dot{\mathcal{L}}_1) = \operatorname{dom}(\mathcal{L}\dot{\mathcal{L}}_2) = \{0\}$, *in particular,* $\operatorname{dom}\dot{\mathcal{L}}_1^2 = \operatorname{dom}\dot{\mathcal{L}}_2^2 = \{0\}$.

If $\operatorname{ran} T = \mathcal{H}$, *then*

(1) *$\dot{\mathcal{L}}_k\mathcal{L}$ is densely defined,*
(2) $(\dot{\mathcal{L}}_k\mathcal{L})^* = \mathcal{L}\dot{\mathcal{L}}_k^*$, $k = 1, 2$,
(3) $\operatorname{dom}(\dot{\mathcal{L}}_1\mathcal{L}) \cap \operatorname{dom}(\dot{\mathcal{L}}_2\mathcal{L}) = \{0\}$ *and*

$$\operatorname{dom}(\dot{\mathcal{L}}_1\mathcal{L}) \dot{+} \operatorname{dom}(\dot{\mathcal{L}}_2\mathcal{L}) = \operatorname{dom}\mathcal{L}^2,$$

(4) *the operator $\mathcal{L}^2 (= T)$ is the Friedrichs extension of $\dot{\mathcal{L}}_1\mathcal{L}$ and $\dot{\mathcal{L}}_2\mathcal{L}$,*
(5) *the Kreĭn extension of the operator $\dot{\mathcal{L}}_j\mathcal{L}$ is $(\dot{\mathcal{L}}_j\mathcal{L})_{\mathrm{K}} = \dot{\mathcal{L}}_j\dot{\mathcal{L}}_j^*$, $j = 1, 2$.*

Proof. Since for all $\varphi \in \operatorname{dom} T_1^{1/2} = \operatorname{dom}\dot{\mathcal{L}}_1$ one has

$$||\dot{\mathcal{L}}_1\varphi||^2 = ||T^{1/2}\varphi||^2 = ||T_1^{1/2}\varphi||^2,$$

we get that $\dot{\mathcal{L}}_1$ is closed operator and the *first representation theorem* [20] leads to equality $\dot{\mathcal{L}}_1^*\dot{\mathcal{L}}_1 = T_1$. Similarly the operator $\dot{\mathcal{L}}_2$ is closed and $\dot{\mathcal{L}}_2^*\dot{\mathcal{L}}_2 = T_2$. Taking into account that $\operatorname{dom} T_k \subset \operatorname{dom} T_k^{1/2}$ and $\operatorname{dom} T_k^{1/2} \cap \operatorname{dom} T = \{0\}$ we obtain $\operatorname{dom} T_k \cap \operatorname{dom} T = \{0\}$, $k = 1, 2$. This means that

$$\operatorname{dom}\dot{\mathcal{L}}_1 \cap \operatorname{dom}\mathcal{L}^2 = \operatorname{dom}\dot{\mathcal{L}}_2 \cap \operatorname{dom}\mathcal{L}^2 = \{0\}.$$

Hence

$$\operatorname{dom}(\dot{\mathcal{L}}_k^*\dot{\mathcal{L}}_k) \cap \operatorname{dom}\mathcal{L}^2 = \{0\}, k = 1, 2. \tag{3.38}$$

The condition $\mathcal{L} \supset \dot{\mathcal{L}}_k$ leads to equality

$$\operatorname{dom}(\dot{\mathcal{L}}_k^*\dot{\mathcal{L}}_k) \cap \operatorname{dom}\mathcal{L}^2 = \operatorname{dom}(\mathcal{L}\dot{\mathcal{L}}_k).$$

Then (3.38) yields $\operatorname{dom}(\mathcal{L}\dot{\mathcal{L}}_k) = \{0\}$, $k = 1, 2$.

Suppose that $\operatorname{ran} T = \mathcal{H}$. Then the operators $\mathcal{L}$, $\dot{\mathcal{L}}_1\mathcal{L}$, and $\dot{\mathcal{L}}_2\mathcal{L}$ are *positive definite*, see Section 1. It follows then that

$$\operatorname{dom}(\dot{\mathcal{L}}_k\mathcal{L}) = \mathcal{L}^{-1}\operatorname{dom}\dot{\mathcal{L}}_k \quad \text{and} \quad (\dot{\mathcal{L}}_k\mathcal{L})(\mathcal{L}^{-1}\varphi) = \dot{\mathcal{L}}_k\varphi,\ \varphi \in \operatorname{dom}\dot{\mathcal{L}}_k,\ k = 1, 2.$$

This yields, that $\operatorname{dom}(\dot{\mathcal{L}}_k\mathcal{L})$ is dense in $\operatorname{dom}\mathcal{L}$ with respect to the graph-norm in $\operatorname{dom}\mathcal{L}$. Hence, the operator $\dot{\mathcal{L}}_k\mathcal{L}$ is densely defined in $\mathcal{H}$ and, moreover, the Friedrichs extensions of $\dot{\mathcal{L}}_k\mathcal{L}$ (for k=1,2) coincide with operator $\mathcal{L}^2$, see Section 2.6 and Theorem 2.7.

Note that $\ker(\dot{\mathcal{L}}_k\mathcal{L})^* = \ker\dot{\mathcal{L}}_k^*$. Therefore, relations

$$\operatorname{dom}\dot{\mathcal{L}}_k^* = \operatorname{dom}\mathcal{L}\dot{+}\ker\dot{\mathcal{L}}_k^* \quad\text{and}\quad \operatorname{dom}(\dot{\mathcal{L}}_k\mathcal{L})^* = \operatorname{dom}\mathcal{L}^2\dot{+}\ker(\dot{\mathcal{L}}_k\mathcal{L})^*$$

lead to the equality $(\dot{\mathcal{L}}_k\mathcal{L})^* = \mathcal{L}\dot{\mathcal{L}}_k^*$, $k = 1, 2$.

The equality $\operatorname{dom}\dot{\mathcal{L}}_1\dot{+}\operatorname{dom}\dot{\mathcal{L}}_2 = \operatorname{dom}\mathcal{L}$ implies that $\operatorname{dom}(\dot{\mathcal{L}}_1\mathcal{L})\cap\operatorname{dom}(\dot{\mathcal{L}}_2\mathcal{L}) = \{0\}$ and

$$\operatorname{dom}(\dot{\mathcal{L}}_1\mathcal{L})\dot{+}\operatorname{dom}(\dot{\mathcal{L}}_2\mathcal{L}) \subseteq \operatorname{dom}\mathcal{L}^2.$$

Let $f \in \operatorname{dom}\mathcal{L}^2 = \operatorname{dom}T$. Then $\mathcal{L}f = T^{1/2}f \in \operatorname{dom}T^{1/2}$. Due to the direct decomposition

$$\operatorname{dom}T_1^{1/2}\dot{+}\operatorname{dom}T_2^{1/2} = \operatorname{dom}T^{1/2},$$

we get $T^{1/2}f = \varphi_1 + \varphi_2$, where $\varphi_k \in \operatorname{dom}T_k^{1/2}$, $k = 1, 2$. The equality $\operatorname{ran}T^{1/2} = \mathcal{H}$ implies that $\varphi_k = T^{1/2}h_k$, $h_k \in \operatorname{dom}T^{1/2}$. Since $\varphi_k = T^{1/2}h_k \in \operatorname{dom}T_k^{1/2}$, $k = 1, 2$, we obtain that $h_k \in \operatorname{dom}(\dot{\mathcal{L}}_k\mathcal{L})$, $k = 1, 2$, and therefore

$$f = T^{-1/2}\varphi_1 + T^{-1/2}\varphi_2 = h_1 + h_2.$$

So we proved

$$\operatorname{dom}\mathcal{L}^2 \subseteq \operatorname{dom}(\dot{\mathcal{L}}_1\mathcal{L})\dot{+}\operatorname{dom}(\dot{\mathcal{L}}_2\mathcal{L}),$$

which implies

$$\operatorname{dom}\mathcal{L}^2 = \operatorname{dom}(\dot{\mathcal{L}}_1\mathcal{L})\dot{+}\operatorname{dom}(\dot{\mathcal{L}}_2\mathcal{L}).$$

Finally, since $\operatorname{ran}\mathcal{L} = \mathcal{H}$, one gets $(\dot{\mathcal{L}}_j\mathcal{L})_{\mathrm{K}} = \dot{\mathcal{L}}_j\dot{\mathcal{L}}_j^*$ for $j = 1, 2$ (see Theorem 2.7). □

Remark 3.30. *Abstract examples of pairs $\langle\mathcal{L}_0, \mathcal{L}\rangle$: $\mathcal{L}_0 \subset \mathcal{L}$, consisting of a densely defined closed and non-negative symmetric operator $\mathcal{L}_0$ and of its non-negative self-adjoint extension $\mathcal{L}$ such that*

(1) $\operatorname{dom}(\mathcal{L}\mathcal{L}_0) = \{0\}$,
(2) *$\mathcal{L}_0\mathcal{L}$ is densely defined, $(\mathcal{L}_0\mathcal{L})^* = \mathcal{L}\mathcal{L}_0^*$, and the operator $\mathcal{L}^2$ is the Friedrichs extension of $\mathcal{L}_0\mathcal{L}$*

are given in [8].

The next two assertions are strengthened versions of the Van Daele–Schmüdgen and Brasche–Neidhardt theorems [40], [36], [11] mentioned in Section 1.

Theorem 3.31. *Let B be unbounded self-adjoint operator in a Hilbert space $\mathcal{H}$. Then there are infinitely many pairs $\langle B_1, B_2\rangle$ of densely defined closed restrictions of B such that*

(1) $\operatorname{dom}B_1 \cap \operatorname{dom}B_2 = \{0\}$;
(2) $\operatorname{dom}B = \operatorname{dom}B_1\dot{+}\operatorname{dom}B_2$;
(3) $\operatorname{dom}B_1 \cap \operatorname{dom}B^2 = \operatorname{dom}B_2 \cap \operatorname{dom}B^2 = \{0\}$;
(4) $\operatorname{dom}(BB_1) = \operatorname{dom}(BB_2) = \{0\}$, *and in particular,* $\operatorname{dom}B_1^2 = \operatorname{dom}B_2^2 = \{0\}$.

If $\operatorname{ran}B = \mathcal{H}$, *then*

(1) *B_kB is densely defined, k=1,2,*
(2) *the operator $B^2 = (\dot{B}_1B)_{\mathrm{F}} = (\dot{B}_2B)_{\mathrm{F}}$ is the Friedrichs extension of the operators $\dot{B}_1B$ and $\dot{B}_2B$,*
(3) $(B_kB)^* = BB_k^*$, $k = 1, 2$,

(4) $\mathrm{dom}\,(B_1B)\cap\mathrm{dom}\,(B_2B)=\{0\}$ *and*
$$\mathrm{dom}\,(B_1B)\dot{+}\mathrm{dom}\,(B_2B)=\mathrm{dom}\,B^2,$$
(5) *the Kreĭn extension of the operator* $\dot{B}_jB$ *is* $(\dot{B}_jB)_{\mathrm{K}}=\dot{B}_j\dot{B}_j^*$, $j=1,2$.

Proof. Let $T=B^2$. Then there exists a pair $\langle T_1,T_2\rangle$ possessing properties 1)–4) mentioned in Theorem 3.20. Now let
$$B_1:=B{\restriction}\,\mathrm{dom}\,T_1^{1/2}\quad\text{and}\quad B_2:=B{\restriction}\,\mathrm{dom}\,T_2^{1/2}.$$
Then $\mathrm{dom}\,B_1\cap\mathrm{dom}\,B_2=\{0\}$. In addition $\mathrm{dom}\,B_1\cap\mathrm{dom}\,B^2=\mathrm{dom}\,B_2\cap\mathrm{dom}\,B^2=\{0\}$. Because $\mathrm{dom}\,B=\mathrm{dom}\,\sqrt{B^2}=\mathrm{dom}\,T^{1/2}$ and $\mathrm{dom}\,T^{1/2}=\mathrm{dom}\,T_1^{1/2}\dot{+}\mathrm{dom}\,T_2^{1/2}$ we get $\mathrm{dom}\,B=\mathrm{dom}\,B_1\dot{+}\mathrm{dom}\,B_2$.

Arguing as in Theorem 3.29 and taking into account that $||B\varphi||^2=||\sqrt{B^2}\varphi||$, $\varphi\in\mathrm{dom}\,B$, we get $B_k^*B_k=T_k$, $k=1,2$. Since $\mathrm{dom}\,T_1\cap\mathrm{dom}\,T=\mathrm{dom}\,T_2\cap\mathrm{dom}\,T=\{0\}$, we get
$$\mathrm{dom}\,(B_1^*B_1)\cap\mathrm{dom}\,B^2=\mathrm{dom}\,(B_2^*B_2)\cap\mathrm{dom}\,B^2=\{0\},$$
Therefore, since $B_1^*\supset B$, $B_2^*\supset B$, and
$$\mathrm{dom}\,(B_k^*B_k)\cap\mathrm{dom}\,B^2=\mathrm{dom}\,(BB_k),\ k=1,2,$$
we obtain the equalities
$$\mathrm{dom}\,(BB_1)=\mathrm{dom}\,(BB_2)=\{0\}.$$
The rest of the theorem can be checked similarly to the proof of the corresponding part of Theorem 3.29. □

Theorem 3.32. *Let $\mathcal{B}$ be a closed densely defined symmetric operator. Then there are infinitely many pairs $\langle\mathcal{B}_1,\mathcal{B}_2\rangle$ of closed densely defined restrictions of $\mathcal{B}$ such that*

(1) $\mathrm{dom}\,\mathcal{B}_1\cap\mathrm{dom}\,\mathcal{B}_2=\{0\}$;
(2) $\mathrm{dom}\,\mathcal{B}=\mathrm{dom}\,\mathcal{B}_1\dot{+}\mathrm{dom}\,\mathcal{B}_2$;
(3) $\mathrm{dom}\,\mathcal{B}_1\cap\mathrm{dom}\,(\mathcal{B}^*\mathcal{B})=\mathrm{dom}\,\mathcal{B}_2\cap\mathrm{dom}\,(\mathcal{B}^*\mathcal{B})=\{0\}$;
(4) $\mathrm{dom}\,(\mathcal{B}^*\mathcal{B}_1)=\mathrm{dom}\,(\mathcal{B}^*\mathcal{B}_2)=\{0\}$, *in particular,* $\mathrm{dom}\,\mathcal{B}_1^2=\mathrm{dom}\,\mathcal{B}_2^2=\{0\}$.

If 0 is for $\mathcal{B}$ the point of the regular type: $||\mathcal{B}f||\ge c||f||$ for some $c>0$ and all $f\in\mathrm{dom}\,\mathcal{B}$, then

(1) $\mathcal{B}_j\mathcal{B}^*$ *is densely defined,*
(2) *the operator $\mathcal{B}\mathcal{B}^*$ is the Friedrichs extension of operators $\mathcal{B}_1\mathcal{B}^*$ and $\mathcal{B}_2\mathcal{B}^*$,*
(3) $(\mathcal{B}_j\mathcal{B})^*=\mathcal{B}\mathcal{B}_j^*$, $j=1,2$,
(4) $\mathrm{dom}\,(\mathcal{B}_1\mathcal{B}^*)\cap\mathrm{dom}\,(\mathcal{B}_2\mathcal{B}^*)=\ker\mathcal{B}^*$ *and*
$$\mathrm{dom}\,(\mathcal{B}_1\mathcal{B}^*)+\mathrm{dom}\,(\mathcal{B}_2\mathcal{B}^*)=\mathrm{dom}\,(\mathcal{B}\mathcal{B}^*),$$
(5) *the Kreĭn extension of the operator $\mathcal{B}_j\mathcal{B}^*$ is the operator $\mathcal{B}_j\mathcal{B}_j^*$, $j=1,2$.*

Proof. Let $\mathcal{B}=UB$ be the polar decomposition of $\mathcal{B}$. Here $B=(\mathcal{B}^*\mathcal{B})^{1/2}$ and U is a partial isometry with $\ker U=\ker\mathcal{B}^*$ and $\mathrm{ran}\,U=\overline{\mathrm{ran}}\,\mathcal{B}$. By Theorem 3.31 there is a pair $\langle B_1,B_2\rangle$ of densely defined closed restrictions of B such that $\mathrm{dom}\,B=\mathrm{dom}\,B_1\dot{+}\mathrm{dom}\,B_2$, $\mathrm{dom}\,B_k\cap\mathrm{dom}\,B^2=\{0\}$, $k=1,2$, and $\mathrm{dom}\,(BB_1)=\mathrm{dom}\,(BB_2)=\{0\}$. Set $\mathcal{B}_k=UB_k$, $k=1,2$. Then clearly $\mathcal{B}_1$ and $\mathcal{B}_2$ are closed densely defined (and symmetric) restrictions of $\mathcal{B}$ with $\mathrm{dom}\,\mathcal{B}_1\cap\mathrm{dom}\,\mathcal{B}_2=\{0\}$, $\mathrm{dom}\,\mathcal{B}=\mathrm{dom}\,\mathcal{B}_1\dot{+}\mathrm{dom}\,\mathcal{B}_2$, and $\mathrm{dom}\,\mathcal{B}_k\cap\mathrm{dom}\,\mathcal{B}^*\mathcal{B}=\{0\}$, $k=1,2$.

Since $\mathcal{B}^* = BU^*$ [20], we have

$$\mathcal{B}^*\mathcal{B}_k = BU^*UB_k = BB_k,\ k = 1, 2.$$

Hence, $\mathrm{dom}\,(\mathcal{B}^*\mathcal{B}_1) = \mathrm{dom}\,(\mathcal{B}^*\mathcal{B}_2) = \{0\}$.

Suppose now that 0 is the point of the regular type for the operator $\mathcal{B}$. Then it is well-known that there exists a self-adjoint extension $\widehat{\mathcal{B}}$ of $\mathcal{B}$ with $\widehat{\mathcal{B}}^{-1} \in \mathbf{B}(\mathcal{H})$, and

$$\begin{aligned}\mathrm{dom}\,\mathcal{B}^* &= \mathrm{dom}\,\widehat{\mathcal{B}} \dot{+} \ker \mathcal{B}^*,\\ \mathrm{dom}\,\mathcal{B}_j^* &= \mathrm{dom}\,\widehat{\mathcal{B}} \dot{+} \ker \mathcal{B}_j^*,\end{aligned}$$

for $j = 1, 2$. Note that

$$\begin{aligned}\mathrm{dom}\,(\mathcal{B}_j\mathcal{B}^*) &= \widehat{\mathcal{B}}^{-1}\mathrm{dom}\,\mathcal{B}_j \dot{+} \ker \mathcal{B}^*,\ j = 1, 2,\\ \mathrm{dom}\,(\mathcal{B}\mathcal{B}_j^*) &= \widehat{\mathcal{B}}^{-1}\mathrm{dom}\,\mathcal{B}_j \dot{+} \ker \mathcal{B}^*,\ j = 1, 2,\\ \mathrm{dom}\,(\mathcal{B}\mathcal{B}^*) &= \widehat{\mathcal{B}}^{-1}\mathrm{dom}\,\mathcal{B} \dot{+} \ker \mathcal{B}^*.\end{aligned}$$

The above equalities and decomposition $\mathrm{dom}\,\mathcal{B} = \mathrm{dom}\,\mathcal{B}_1 \dot{+} \mathrm{dom}\,\mathcal{B}_2$ leads to $\mathrm{dom}\,(\mathcal{B}_1\mathcal{B}^*) \cap \mathrm{dom}\,(\mathcal{B}_2\mathcal{B}^*) = \ker \mathcal{B}^*$ and $\mathrm{dom}\,(\mathcal{B}_1\mathcal{B}^*) + \mathrm{dom}\,(\mathcal{B}_2\mathcal{B}^*) = \mathrm{dom}\,(\mathcal{B}\mathcal{B}^*)$.

Now we show that $\mathrm{dom}\,(\mathcal{B}_k\mathcal{B}^*)$ is dense in $\mathrm{dom}\,\mathcal{B}^*$ with respect to the graph inner-product in $\mathrm{dom}\,\mathcal{B}^*$. If for some $h \in \mathrm{dom}\,\mathcal{B}^*$ one has

$$(\mathcal{B}^*(\widehat{\mathcal{B}}^{-1}\varphi_1 + \varphi_0), \mathcal{B}^*h) + (\widehat{\mathcal{B}}^{-1}\varphi_1 + \varphi_0, h) = 0$$

for all $\varphi_1 \in \mathrm{dom}\,(\mathcal{B}_1\mathcal{B}^*)$ and for all $\varphi_0 \in \ker \mathcal{B}^*$, then $h = \mathcal{B}g$ and

$$(\varphi_1, \mathcal{B}\mathcal{B}^*g + g) = 0 \text{ for all } \varphi_1 \in \mathrm{dom}\,\mathcal{B}_1.$$

Since $\mathrm{dom}\,\mathcal{B}_1$ is dense in $\mathcal{H}$, we get $g = 0$. Thus $\mathrm{dom}\,(\mathcal{B}_1\mathcal{B}^*)$ and similarly $\mathrm{dom}\,(\mathcal{B}_2\mathcal{B}^*)$ are dense in $\mathrm{dom}\,\mathcal{B}^*$ with respect to the graph inner-product in $\mathrm{dom}\,\mathcal{B}^*$. This is equivalent to the fact that $\mathrm{dom}\,(\mathcal{B}_j\mathcal{B}^*)$ is dense in $\mathcal{H}$ and that the Friedrichs extension of $\mathcal{B}_j\mathcal{B}^*$ is the operator $\mathcal{B}\mathcal{B}^*$ for any of $j = 1, 2$, i.e., $(\mathcal{B}_j\mathcal{B}^*)_{\mathrm{F}} = \mathcal{B}\mathcal{B}^*$.

Since

$$\begin{cases}\mathrm{dom}\,(\mathcal{B}_1\mathcal{B}^*) \ni f = \widehat{\mathcal{B}}^{-1}\varphi_1 + \psi_0,\ \varphi_1 \in \mathrm{dom}\,\mathcal{B}_1,\ \psi_0 \in \ker \mathcal{B}_1^*,\\ (\mathcal{B}_1\mathcal{B}^*)f = \mathcal{B}_1\varphi_1\ ,\end{cases}$$

for any $x \in \mathrm{dom}\,(\mathcal{B}_1\mathcal{B}^*)^*$ we get

$$(\mathcal{B}_1\varphi_1, x) = (\widehat{\mathcal{B}}^{-1}\varphi_1 + \psi_0, (\mathcal{B}_1\mathcal{B}^*)^*x)$$

for all $\varphi_1 \in \mathrm{dom}\,\mathcal{B}_1$ and all $\psi_0 \in \ker \mathcal{B}_1^*$. This implies that

$$\begin{aligned}&(\mathcal{B}_1\mathcal{B}^*)^*x = \mathcal{B}g,\ g \in \mathrm{dom}\,\mathcal{B}_1,\\ &x \in \mathrm{dom}\,\mathcal{B}_1^*,\ \mathcal{B}_1^*x = g.\end{aligned}$$

Therefore, if $x \in \mathrm{dom}\,(\mathcal{B}_1\mathcal{B}^*)^*$, then $x \in \mathrm{dom}\,\mathcal{B}\mathcal{B}_1^*$ and $(\mathcal{B}_1\mathcal{B}^*)^*x = \mathcal{B}\mathcal{B}_1^*x$, i.e.,

$$(\mathcal{B}_1\mathcal{B}^*)^* \subseteq \mathcal{B}\mathcal{B}_1^*.$$

On the other hand one has

$$\mathcal{B}\mathcal{B}_1^* \subseteq (\mathcal{B}_1\mathcal{B}^*)^*.$$

Thus, $(\mathcal{B}_1\mathcal{B}^*)^* = \mathcal{B}\mathcal{B}_1^*$. Similarly we obtain $(\mathcal{B}_2\mathcal{B}^*)^* = \mathcal{B}\mathcal{B}_2^*$. Applying Theorem 2.7 we get that

$$(\mathcal{B}_j\mathcal{B}^*)_{\mathrm{K}} = \mathcal{B}_j\mathcal{B}_j^*,\ j = 1, 2.$$

This completes the proof. □

Combining Theorem 3.20, Corollary 3.24, and Theorems 3.26, 3.32 we can summarise them as the following statement.

Theorem 3.33. *Let $\mathcal{B}$ be a closed densely defined symmetric operator. Then*

(1) *for each natural number $n \in \mathbb{N}$ there exists n closed densely defined restrictions $\{\mathcal{B}_k\}_{k=1}^n$ of the operator $\mathcal{B}$ such that:*
   (a) $\operatorname{dom}\mathcal{B}_k \cap \operatorname{dom}\mathcal{B}_j = \{0\}$, $k \neq j$,
   (b) $\operatorname{dom}\mathcal{B}_1 \dot{+} \operatorname{dom}\mathcal{B}_2 \dot{+} \cdots \dot{+} \operatorname{dom}\mathcal{B}_n = \operatorname{dom}\mathcal{B}$,
   (c) $\operatorname{dom}\mathcal{B}_k \cap \operatorname{dom}(\mathcal{B}^*\mathcal{B}) = \{0\}$ *for each* $k = 1, 2, \ldots, n$,
   (d) $\operatorname{dom}(\mathcal{B}^*\mathcal{B}_k) = \{0\}$ *for each* $k = 1, 2, \ldots, n$,

(2) *there exists two infinite sequences $\{\mathcal{B}_{1,j}\}_{j\in\mathbb{N}}$ and $\{\mathcal{B}_{2,j}\}_{j\in\mathbb{N}}$ of closed densely defined restrictions of the operator $\mathcal{B}$ such that*
   (a) $\mathcal{B} \supset \mathcal{B}_{1,1} \supset \mathcal{B}_{1,2} \supset \cdots \supset \mathcal{B}_{1,j} \supset \cdots$,
   (b) $\mathcal{B}_{2,1} \subset \mathcal{B}_{2,2} \subset \cdots \subset \mathcal{B}_{2,j} \subset \cdots \subset \mathcal{B}$,
   (c) $\bigcap_{j\in\mathbb{N}} \operatorname{dom}\mathcal{B}_{1,j} = \{0\}$,
   (d) $\operatorname{dom}\mathcal{B}_{1,j} \dot{+} \operatorname{dom}\mathcal{B}_{2,j} = \operatorname{dom}\mathcal{B}$,
   (e) $\operatorname{dom}(\mathcal{B}^*\mathcal{B}_{1,j}) = \{0\}$, $\operatorname{dom}(\mathcal{B}^*\mathcal{B}_{2,j}) = \{0\}$ *for each* $j \in \mathbb{N}$.

4. Acknowledgements

Yu.A. was supported by Aix-Marseilles University during his visit the Laboratoire d'Analyse, Topologie, Probabilités /Institut de Mathématiques de Marseille (UMR 7353).

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